Starred exercises are optional.

1. Give a closed-form expression for each of the following recurrences (1), (2), and (3), for $n > 1$:

$$
\begin{aligned}
T_1(n) &= 3 \cdot T_1\left(\frac{n}{3}\right) + n^2 && (1)\\
T_2(n) &= 100 \cdot T_2\left(\frac{n}{4}\right) + 5 \cdot n^3 && (2)\\
T_3(n) &= T_3\left(\frac{n}{5}\right) && (3)
\end{aligned}
$$

2. For the recurrence $T(n) = T(n-1) + 2 \cdot n - 1$ if $n > 0$ and 0 otherwise, specifying some function $\mathbb{N} \to \mathbb{N}$ whose closed form is to be determined:
   - substitute the equation a number of times in itself, describe the pattern you find, and determine the number of iterations k after which the base case is reached.
   - compute the values of T on inputs $0,\ldots,4$.
   - based on the previous items, guess a closed-form solution, i.e. some function $f : \mathbb{N} \to \mathbb{N}$, for the recurrence and verify its correctness by substitution.

3. A famous algorithm for finding the median of a list is the so-called median-of-medians algorithm. Let the recurrence for its time-complexity be given by $T(n) = c \cdot n$ for $1 \leq n < 21$, and otherwise by:

$$
T(n) = T\left(\left\lceil \frac{2}{10} \cdot n \right\rceil\right) + T\left(\left\lceil \frac{7}{10} \cdot n \right\rceil\right) + c \cdot n
$$

   Argue why a closed-form for this recurrence cannot be found by the Master theorem. Then prove that in fact $\forall n > 0,\ T(n) \leq 12 \cdot c \cdot n$, by well-founded induction on n ordered by $<$. Conclude that the median-of-medians algorithm is linear, i.e. in $\mathrm{O}(n)$.

4∗) The Fibonacci numbers are specified by the *recurrence* on slide 8 of the lecture of week 11. Verify that their *closed-form* definition, given later on the same slide, is a solution for the recurrence, by *substitution*. Is induction needed to prove this?

5∗) Guess/find a closed-form for the function $\mathbb{N} \to \mathbb{N}$ specified by $T(n) = T(n-1) + 3 \cdot n^2 - 3 \cdot n - 1$ if $n > 1$ and 0 otherwise, and verify the guess is correct by substitution. *Hint*: Guess it is a polynomial.