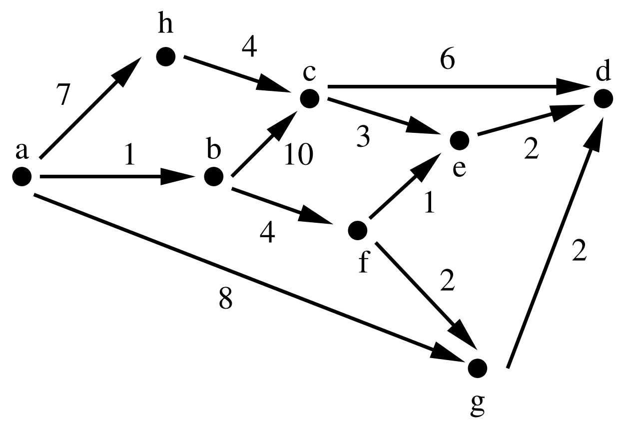
Discrete Structures/Mathematics WS 2019/2020 LVA 703070/703015

EXAM Test. (Each exercise corresponds to 1 cross in the PS) January 22nd, 2020

This exam consists of three regular exercises (1–3). The time available is 1 hour and 45 minutes (105 minutes). The available points for each item are written in the margin. There are 60 points in total for the regular exercises. In addition, there is a bonus exercise (4∗) worth 20 points. You need at least 30 points to pass.

1 Let the weighted directed graph G with set of vertices $V = \{a, b, c, d, e, f, g, h\}$ be:



[5] (a) Give a matrix representation M of G.

[5] (b) Compute a shortest path π from node a to node d using the shortest path algorithm for dags, giving at least two intermediate stages of the algorithm.

[5] (c) Let R be the binary relation R on V underlying graph G, and let the function $f : V \to \mathbb{N}$ be defined by $f(v) = \Sigma_{wRv} f(w)$, i.e. the f-value of node v is the sum of the f-values of all nodes R-related to v. Give R (as set of pairs), explain why f is a well-defined function, and compute $f(e)$.

[5] (d) Let U be the *undirected* graph associated to G, obtained by forgetting the direction of the edges of G. Compute a minimal spanning tree T of U using Kruskal's algorithm, giving at least two intermediate stages of the algorithm.

[5] **2** (a) Let $A = \{a, b, c, d, e\}$ and $B = \{0, 1\}$. Compute the number s of subsets of A, the number t of subsets of size 2 of A, and the number f of (total) functions from A to B.

[5] (b) Let $p = 17$, $q = 12$, $a = 3$ and $b = 4$. Compute the inverse p' of p modulo q, and an x such that $x \equiv a \pmod{p}$ and $x \equiv b \pmod{q}$. Indicate which algorithm(s) you use and give at least two intermediate stages for both computations.

[5] (c) Suppose the complexity T of some algorithm A, as a function of the size n of its input, is given by $T(n) = T\left(\frac{n}{2}\right) + 4 \cdot n$ if $n > 4$, and 5 otherwise. Determine a closed-form expression e such that $T(n) \in \Theta(e)$, and explain what the latter notation means.

[5] (d) Show that the set $\{M\#x \mid \text{TM } M \text{ rejects input } x\}$ is not recursive, where you may assume that (the code of) M and x are bit-strings.

[20] $\boxed{3}$ Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

statement
$(A^\ell)_{ii}$ is the number of cycles of length ℓ on node v_i, for A the adjacency matrix of a directed graph with nodes $v_1, \ldots, v_n$.
Well-founded orders are closed under union.
For a partial order, every minimal element is least.
The specification $f(0) = 1$ and $f(n) = f(f(n-1)) - 1$ if $n > 0$, defines a function $f : \mathbb{N} \to \mathbb{N}$, with $-$ the usual subtraction of integers.
By inclusion/exclusion, $\#(\bigcup_{i=1}^k A_i) \leq \sum_{I \subseteq \{1,\ldots,k\},\ \#(I) \text{ odd}} \#(\bigcap_{i \in I} A_i)$, for sets $(A_i)_{1 \leq i \leq k}$.
If $c \cdot a \equiv c \cdot b \bmod n$, then $a \equiv b \bmod n$, for all natural numbers $a, b, c, n \geq 2$.
For all functions f, g on $\mathbb{N}$, $f \in O(g)$ or $g \in O(f)$ or both.
If there is an injective function from A to the set $\{n \in \mathbb{N} \mid n \geq 10\}$, then A is finite.
If $L = L(M)$ for some TM M, then L or $\sim L$ is recursive.
For all DFAs A there exist strings x, y and a state q such that $\hat{\delta}(q, x) = \hat{\delta}(q, y)$ and $x \neq y$.

$\boxed{4*}$

[10] (a) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(0) = 0$ and and $f(n) = f(n-1) + 3 \cdot n \cdot (n-1) + 1$ if $n \geq 1$. Show that $\forall n \in \mathbb{N},\ f(n) = n^3$.

[10] (b) Prove that there is a bijection between the set $\mathbb{N}$ of natural numbers and the set P of palindromes over $\{0, 1\}$, i.e. P contains all bit-strings x that are the same as their reverse.