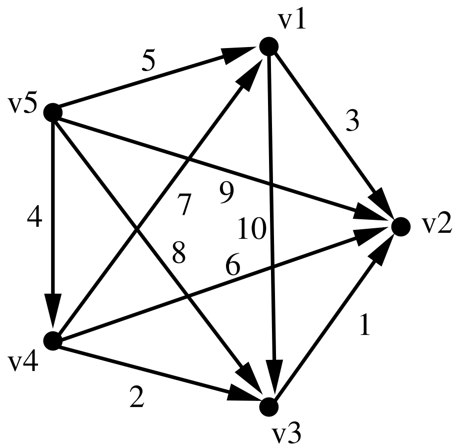
universität innsbruck

Discrete Structures/Mathematics | WS 2019/2020 | LVA 703070/703015

EXAM 1 | January 29th, 2020

This exam consists of <u>three</u> regular exercises (1–3) each worth 20 points. The time available is 1 hour and 45 minutes (105 minutes). The available points for each item are written in the margin. There are 60 points in total for the regular exercises. In addition, there are bonus exercises (4∗, 5∗) each worth 15 points. You need at least 30 points to pass.

1 Let the weighted directed graph G with set of vertices $V = \{v1, v2, v3, v4, v5\}$ be:



Let R be the relation on vertices of graph G.

[6] (a) Give the Hasse diagram of R and show that R is a strict order.

[7] (b) Compute a shortest path from $v5$ to $v2$ in G using an algorithm of your preference. Indicate the algorithm used and give at least 2 intermediate stages of the algorithm.

[7] (c) Write a recursive specification for the function f mapping a node v in G to the sum of the weights of all paths from v to $v2$. For instance, evaluating $f(v1)$ should result in 14. Your specification should not use any concrete weights in G, but be stated in terms of the function w assigning weights to edges. Using your recursive specification, stepwise evaluate $f(v4)$.

2

[5] (a) Compute $7^{100} \bmod 11$, and compute an inverse of 12 modulo 17 and verify that it indeed is inverse. Justify the steps taken to compute the results.

[5] (b) Let R be the relation on pairs of natural numbers defined by: $(n, m)\ R\ (n', m')$ if $n + m' = n' + m$. For instance, $(5, 3)\ R\ (2, 0)$ since $5 + 0 = 5 = 2 + 3$, but not $(5, 3)\ R\ (0, 2)$ since $5 + 2 = 7 \neq 3 = 0 + 3$. Show that R is an equivalence relation, and give 4 elements of the R-equivalence class $[(0, 2)]$ of $(0, 2)$.

[5] (c) Suppose the complexity T of some algorithm A, as a function of the size n of its input, is an increasing function that satisfies $T(n) = 4 \cdot T\left(\frac{n}{2}\right) + 2 \cdot n^2$ if $n = 2^k$ for some positive natural number k, and 2 if $n = 1$. Determine a closed-form expression e such that $T(n) \in \Theta(e)$, and explain what the latter notation means.

[5] (d) Show that the set $\{M\#x \mid \text{TM } M \text{ loops on input } x\}$ is not recursive, where you may assume that (the code of) M and x are bit-strings.

[20] **3** Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

statement
For every partial order $\leq$ on A, its complement $\sim(\leq) = (A \times A) - \leq$ is a partial order.
For every regular language L over Σ, its complement $\sim L = \Sigma^* - L$ is regular.
For every recursively enumerable language L over Σ, its complement $\sim L = \Sigma^* - L$ is recursively enumerable.
For every set B and countable subset $A \subset B$, its complement $\sim A = B - A$ is countable.
The specification $f(x) = f(x) \cdot 2$ defines a function $f : \mathbb{N} \to \mathbb{N}$.
Let $\mathbb{N}_2 = \mathbb{N} - \{0, 1\}$. For every $n \in \mathbb{N}_2$, n is prime iff n is minimal w.r.t. divisibility in $\mathbb{N}_2$.
The function mapping n to the pair $(n \bmod 10, n \bmod 7)$ is a bijection between $\{0, \ldots, 69\}$ and the set of pairs of natural numbers $\{(x, y) \mid 0 \leq x < 10, 0 \leq y < 7\}$.
For all natural numbers a, b, if $\gcd(a, b) = 1$, then $a^1, a^2, \ldots, a^b$ are all distinct modulo b.
If $f : A \to B$, $g : B \to C$, $h : C \to A$ are injective functions, then A and B are equinumerous.
For every DFA A and each of its states q, there exist strings x, y such that $\hat{\delta}(q, x) = \hat{\delta}(q, y)$ and $x \neq y$.

4∗ Let $\sqsubseteq$ be the lexicographic order on pairs of natural numbers, with $\sqsubset$ its strict part. For instance, $(3, 5) \sqsubset (5, 3) \sqsubset (5, 5) \sqsubseteq (5, 5)$. Let R be the relation on pairs of natural numbers defined in Exercise 2(b).

[5] (a) Show that $\sqsubseteq$ is a total relation on pairs of natural numbers.

[5] (b) Show that each R-equivalence class has a unique $\sqsubseteq$-minimal element, give such $\sqsubseteq$-minimal elements for $[(3, 5)]$, $[(5, 3)]$, and $[(5, 5)]$, and argue that taking $\sqsubseteq$-minimal elements yields a system of representatives of R.

[5] (c) Show that there is a bijection between the set of integers $\mathbb{Z}$ and the set of equivalence classes of R.

5∗

[5] (a) Show that for any n, there is a dag having n nodes and $\frac{n \cdot (n-1)}{2}$ edges. (Cf. the graph G of Exercise 1, which has 5 nodes and $\frac{5 \cdot 4}{2} = 10$ edges),

[5] (b) Show that the number given in the previous item is maximal, i.e. show that there are no dags having n nodes and more than $\frac{n \cdot (n-1)}{2}$ edges.

[5] (c) Give a *maximal* spanning tree T of the undirected graph U corresponding to the directed graph G of Exercise 1, i.e. a spanning tree having maximal weight among all spanning trees. Argue why your tree T is indeed a maximal spanning tree.