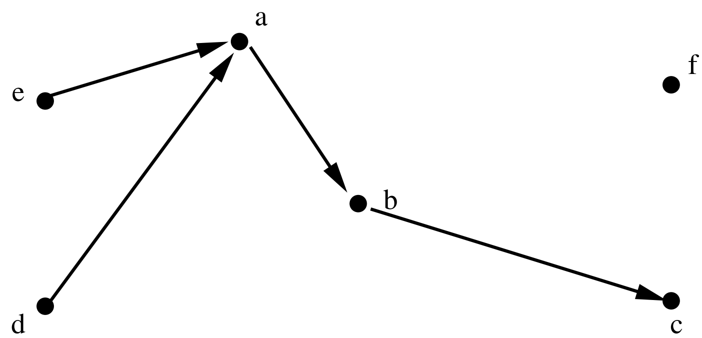
Discrete Structures/Mathematics WS 2019/2020 LVA 703070/703015

EXAM 2 February 28th, 2020, 9:00–10:45

This exam consists of three regular exercises (1–3) each worth 20 points. The time available is 1 hour and 45 minutes (105 minutes). The available points for each item are written in the margin. There are 60 points in total for the regular exercises. In addition, there are bonus exercises (4∗, 5∗) each worth 15 points. You need at least 30 points to pass.

1 Let the Hasse diagram of the partial order $\leq$ on $\{a, b, c, d, e, f\}$ be given by the dag G:



so that, e.g., $a \leq c$ but not $c \leq a$.

[6] (a) Give the partial order $\leq$, as set of pairs, and indicate how it is obtained from G.

[6] (b) Give 3 distinct topological sortings of G, and indicate how they are obtained from G.

[8] (c) Write a recursive specification for the function r mapping a node v in G to the set of all nodes reachable from v in G. For instance, $r(a) = \{a, b, c\}$. Your specification should not use any concrete edges or nodes in G, only general ones. Argue why your specification is well-defined, and using it, stepwise evaluate $r(e)$.

2 [5] (a) Compute $13^{14} \bmod 15$, and compute an inverse of 13 modulo 14 and verify that it indeed is inverse. Justify the steps taken to compute the results.

[5] (b) Let R, S be the relations on pairs of real numbers defined by $(x, y)\ R\ (x', y')$ if $x^2 + y^2 = x'^2 + y'^2$ respectively $(x, y)\ S\ (x', y')$ if $x^2 = x'^2$. Show that R and S are equivalence relations, and draw the R- and S-equivalence classes of $(0, 0)$, $(1, 0)$, and $(2, 0)$ in the usual two-dimensional cartesian coordinate system.

[5] (c) Suppose the complexity T of some algorithm A, as a function of the size n of its input, is an increasing function that satisfies $T(n) = 5 \cdot T\left(\frac{n}{2}\right) + 2 \cdot n^2$ if $n = 2^k$ for some positive natural number k, and 2 if $n = 1$. Determine a closed-form expression e such that $T(n) \in \Theta(e)$, and explain what the latter notation means.

[5] (d) Show that the language $L = \{M\#x \mid \text{TM } M \text{ never moves to the left on input } x\}$ is recursive.

[20] **3** Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

statement
If there is an injection $f : A \to B$ for $A \supset B$ (strict superset), then A has a countably infinite subset.
Taking the symmetric closure of the transitive closure of R yields the same relation as taking the transitive closure of the symmetric closure of R, for every binary relation R.
If a partial order $\leq$ has a least element, then its strict part $<$ is well-founded.
There are uncountably many countable sets.
The specifications $f(n) = 2 \cdot f(n-1) + f(n-2)$ if $n \geq 2$, and n otherwise, and $g(n) = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$ define functions that are the same: $\forall n \in \mathbb{N}, f(n) = g(n)$ (where $0 \in \mathbb{N}$).
$\frac{\text{lcm}(2^5 \cdot 17^2 \cdot 4, 7)}{\text{gcd}(2^8 \cdot 17 \cdot 49, 2^6 \cdot 17^2)} = 34$.
If there are 150 persons in a room, then there are at least 2 among them having the same age (in years).
For all natural numbers a, b, if $\gcd(a, b) = 1$, then $a^1, a^2, \ldots, a^{b-1}$ are all distinct modulo b.
There is a countable set A such that A is equinumerous to the set $A \to A$ of all total functions from A to A.
For every DFA A and each pair of states p, q of A, there exists a string x, such that $\hat{\delta}(p, x) = q$ or $\hat{\delta}(q, x) = p$.

[8] **4∗** (a) Show that the sets of equivalence classes for R and S as in Exercise 2(b), are equinumerous and uncountable.

[7] (b) Let $p = 15$, $q = 13$, $a = 4$ and $b = 5$. Compute an x such that $x \equiv a \pmod{p}$ and $x \equiv b \pmod{q}$. Indicate which algorithm(s) you use and give at least two intermediate stages for your computation(s).

5∗ Consider the following specification of M on $\mathbb{N}$ (where $\mathbb{N}$ includes 0): $M(n) = M(M(n+11))$ if $n \leq 100$ and $n - 10$ otherwise.

[4] (a) Compute $M(99)$.

[11] (b) Prove that the above specification specifies a function M from $\mathbb{N}$ to $\mathbb{N}$, by induction. Indicate the type of induction you employ.