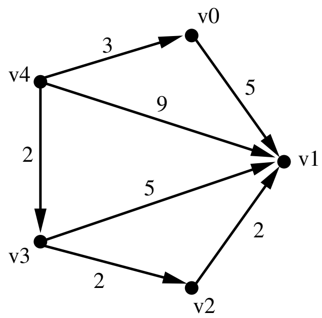
universität innsbruck

Discrete Structures/Mathematics WS 2019/2020 LVA 703070/703015

EXAM 3 September 25th, 2020, 9:00–10:45

This exam consists of <u>three</u> regular exercises (1–3) each worth 20 points. The time available is 1 hour and 45 minutes (105 minutes). The available points for each item are written in the margin. There are 60 points in total for the regular exercises. In addition, there is a bonus exercise (4∗) worth 17 points. You need at least 30 points to pass.

$\boxed{1}$ Let the graph G with nodes $\{v0, v1, v2, v3, v4\}$ be given by:



(a) Compute the length of the shortest path from $v4$ to $v1$ by means of
[8] Floyd's algorithm, giving the matrices for each of its 6 stages.

(b) Explain what a topological sorting of the nodes of graph G is, and argue
[5] that G has exactly 3 topological sortings.

(c) Write a recursive specification for the function r mapping a natural number n and a node v in a directed graph to the set of all nodes reachable from v by a path having exactly n edges. For instance, in G we
[7] have $r(0, v0) = \{v0\}$ and $r(1, v4) = \{v0, v1, v3\}$ and $r(2, v2) = \emptyset$. Argue why your specification is well-defined, and stepwise evaluate $r(2, v4)$ in G using your specification.

$\boxed{2}$ (a) Compute 15^{14} mod 13, and compute an inverse of 15 modulo 13 and
[5] verify that it indeed is inverse. Justify the steps taken.

(b) Let R be the relation on pairs of positive natural numbers defined by $(n, m)\ R\ (n', m')$ if $n \cdot m' = n' \cdot m$. Show R is an equivalence relation
[5] and give two pairs R-related to $(3, 5)$.

(c) Let $p = 9$, $q = 8$, $a = 7$ and $b = 6$. Compute an x such that $x \equiv a$
[5] $\pmod{p}$ and $x \equiv b \pmod{q}$. Indicate which algorithm(s) you use and give at least two intermediate stages for your computation(s).

[5] (d) Show that the language $L = \{M\#x \mid \text{on input } x \text{ TM } M \text{ moves the head at some moment to the 10th position on the tape}\}$ is recursive.

[20] $\boxed{3}$ Determine whether the following statements are true or false. Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

statement
Given two disjoint sets A, B, there is no injection from $A \cup B$ into A.
A binary relation R on A that is reflexive, transitive and has the property that for all elements $a, b, c \in A$, if $a\ R\ b$ and $a\ R\ c$ then $b\ R\ c$, is an equivalence relation.
Suppose the partial order $\leq$ on A is total, i.e. for all $a, b \in A$ we have $a \leq b$ or $b \leq a$ (possibly both). Then an element is $\leq$-least iff it is $\leq$-minimal.
Let R be the relation between real numbers that relates x and y if $\lvert x - y \rvert < 5$. Then R is symmetric and transitive.
Let f and g be functions on the natural numbers specified by $f(0) = 0$, $f(1) = 1$, and $f(n) = f(n-2) + f(n-1)$ otherwise (the Fibonacci numbers), and $g(0) = 1$, $g(1) = 2$, and $g(n) = g(n-2) \cdot g(n-1)$ otherwise. Then for all n, $2^{f(n)} = g(n)$.
$\frac{\mathrm{lcm}(5^2 \cdot 17^2 \cdot, 7 \cdot 5^4)}{\gcd(5^7 \cdot 17^3 \cdot 49, 5^4 \cdot 17^2)} = 7$.
A relation R is well-founded iff its transitive closure R^+ is well-founded.
Consider the following list of 10 natural numbers 12345, 56434, 88888, 98430, 76455, 28451, 17154, 98342, 94563, 33223. Among these numbers there are numbers x, y with $x \neq y$ whose difference $x - y$ is divisible by 7.
Let f be an arbitrary function from N to N, where $N = \{1, \ldots, 117\}$. Then the infinite sequence s given by $0, f(0), f(f(0)), f(f(f(0))), \ldots$ becomes repetitive. That is, s has shape $s_0, \ldots, s_{k-1}, s_k, \ldots, s_{k+\ell-1}, s_k, \ldots, s_{k+\ell-1}, s_k, \ldots, s_{k+\ell-1}, \ldots$ for some k (length of initial part) and $\ell > 0$ (length of cycle), i.e. $s_i = s_{\ell+i}$ for all $i \geq k$.
The sets $\mathbb{Z}$ (the integers) and $\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$ (the real numbers between -1 and 1) are equinumerous.

$\boxed{4*}$

[5] (a) Let R_0 be specified as R in Exercise 2(b), but now for the natural numbers *including* 0. Give an example that shows R_0 is not an equivalence relation.

[6] (b) Suppose the complexity T of some algorithm A, as a function of the size n of its input, is an increasing function that satisfies $T(n) = 5 \cdot T\left(\frac{n}{2}\right) + 2 \cdot n^3$ if $n = 2^k$ for some positive natural number k, and 2 if $n = 1$. Determine a closed-form e such that $T(n) \in \Omega(e)$, and explain the Ω-notation.

[6] (c) Show that if there is a shortest path from from a node v to a node w in an integer-weighted directed graph G, with G having n nodes, then there is also a shortest path from v to w in G having fewer than n edges.