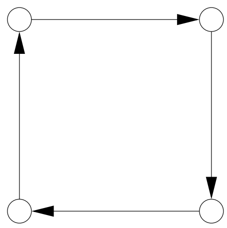
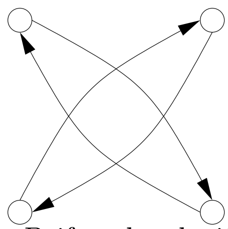
1) • We may draw G as



Enumerating vertices and edges clockwise, starting from the top–right vertex, as v_i respectively e_i, the graph may be represented as $(\{v_i \mid 0 \leq i < 4\}, \{(v_i, v_{i+1 \pmod 4}) \mid 0 \leq i < 4\})$ with $src((v, v')) = v$ and $tgt((v, v')) = v'$.

• $B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ or in a drawing  . There is an edge in the graph represented by the adjacency matrix B if and only if there is a path of length 2 in the graph represented by A; this is because $A_{ik}A_{kj} = 1$ iff there is a path of shape $((v_i, v_k), (v_k, v_j))$, so that $\max_{k=1}^{4} A_{ik}A_{kj} = 1$ iff there is such a path for some k.

Generalising this to arbitrary n, we see that there is an edge (v, w) in the n-fold 'matrix product' iff there is a path from v to w of length n in A. To make this hold also in case $n = 0$, we define the 0-fold 'matrix product' to be the identity matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

2) • Let p be the shortest path from a to c taken. That the path goes via b means p is the composition of paths q from a to b and r from b to c. By definition of weight/length, then $w(p) = w(q) + w(r)$.

For a proof by contradiction, suppose there were a shorter path from a to b, say path q' with weight $w(q') < w(q)$. Then the composition p' of q' and r would be a path from a to c having weight $w(p') = w(q') + w(r) < w(q) + w(r) = w(p)$, contradicting our assumption that p is the shortest path from a to c.

• For a proof by contradiction, suppose G has n nodes and that $p = (e_0, \ldots, e_{\ell-1})$ of length $\ell \geq n$ is a shortest path from a to b in G. By definition of path there are nodes $v_0, \ldots, v_\ell$ such that $v_0 = a$, $v_\ell = b$, and for all $0 \leq i < \ell$, $v_i = src(e_i)$ and $tgt(e_i) = v_{i+1}$. Because the sequence $v_0, \ldots, v_\ell$ of nodes has length $\ell + 1 > \ell \geq n$ and there are only n nodes, at least one node must occur more than once in it, i.e. $v_i = v_j$ for some $0 \leq i < j \leq \ell$. But then the path $p' = (e_0, \ldots, e_{i-1}, e_j, \ldots, e_{\ell-1})$ is also a path from a to b (note that indeed $tgt(e_{i-1}) = v_i = v_j = src(e_j)$), and shorter than p since $w(p') = i + (\ell - j) = \ell - (j - i) < \ell = w(p)$, contradicting our supposition that p is the shortest such.

3) We compute for B:

$$\begin{pmatrix} 0 & 1 & 3 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & \infty \\ 1 & \infty & 5 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 3 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & \infty \\ 1 & \color{red}{2} & \color{red}{4} & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & \color{red}{2} & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & \infty \\ 1 & 2 & \color{red}{3} & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & \infty \\ 1 & 2 & 3 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & \infty \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

and for B':

$$\begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & 5 & 0 & 1 \\ 1 & 3 & \infty & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & 5 & 0 & 1 \\ 1 & \color{red}{2} & \infty & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & 5 & 0 & 1 \\ 1 & 2 & \infty & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & 5 & 0 & 1 \\ 1 & 2 & \infty & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \color{red}{2} & \color{red}{3} & 0 & 1 \\ 1 & 2 & \infty & 0 \end{pmatrix}$$

We obtain the same distances. More precisely, just like B is obtained by 'rotating' the columns of B left and the rows up, the same holds for their distance matrices resulting from Floyd. (But note that this, in general, does not hold at the various intermediate stages!) This holds by correctness of Floyd's algorithm, since distances are unique, for any pair of nodes.

4∗)

- Since an n by n matrix over $\{0, 1\}$ has n^2 entries and each entry may take 2 values, the number of such matrices is 2^{n^2}. Fixing the natural order on the nodes $\{1, \ldots, n\}$, every digraph on these nodes can be made to correspond to an adjacency matrix, unique for the given order, and vice versa. Hence there are 2^{n^2} such digraphs.
- Consider the Haskell program:

```
digraphs :: Int -> Int
digraphs n = 2 ^ n ^ 2
subsets :: Int -> [[Int]]
subsets 0 = [[]]
subsets n = s ++ map ([n]++) s
  where s = subsets (n-1)
digraphslist :: Int -> [[(Int,Int)]]
digraphslist 0 = [[]]
digraphslist n = [ (map (\z->(n,z)) xl)++(map (\z->(z,n)) xr)++y |
  xl <- subsets n, xr <- subsets (n-1), y <- digraphslist (n-1) ]
```

Although the function `digraphs` produces all the graphs for a given number of nodes implicitly, by returning the number of such, `digraphslist` produces them explicitly as lists of edges (still leaving the set of vertices implicit). Testing it in ghci yields:

```
*Main> digraphs 2
16
*Main> digraphslist 2
[[],[(1,1)],[(1,2)],[(1,2),(1,1)],[(2,1)],[(2,1),(1,1)],[(2,1),(1,2)],[(2,1),
(1,2),(1,1)],[(2,2)],[(2,2),(1,1)],[(2,2),(1,2)],[(2,2),(1,2),(1,1)],[(2,2),(
2,1)],[(2,2),(2,1),(1,1)],[(2,2),(2,1),(1,2)],[(2,2),(2,1),(1,2),(1,1)]]
*Main> length it
16
```

For 3 nodes there are already 512 graphs and for 4 we have 65536 of them.