1) The Master theorem applies to each of the three recurrences:

   a) since for $a = 3$, $b = 3$, $s = 2$ we have $a = 3 < 9 = 3^2 = b^s$, its 3rd case applies, so $T_1(n) \in \Theta(n^s) = \Theta(n^2)$;

   b) since for $a = 100$, $b = 4$, $s = 3$ we have $a = 100 > 64 = 4^3 = b^s$, its 1st case applies, so $T_2(n) \in \Theta(n^{\log_b a}) \approx \Theta(n^{3.32})$;

   c) since for $a = 1$, $b = 5$, $s = 0$ we have $a = 1 = 5^0 = b^s$, its 2nd case applies, so $T_3(n) \in \Theta(n^s \cdot \log n) = \Theta(\log n)$.

2) For the first recurrence we proceed as:

   a)

$$\begin{aligned} T(n) &= T(n-1) + 2 \cdot n - 1 \\ &= T(n-2) + 2 \cdot (n-1) - 1 + 2 \cdot n - 1 \\ &= T(n-3) + 2 \cdot (n-2) + 2 \cdot (n-1) + 2 \cdot n - 3 \\ &= \dots \\ &= T(0) + 2 \cdot \sum_{i=0}^{n} i - n \end{aligned}$$

   The recursion stops after $n$ recursive calls.

   b) $T(0) = 0$, $T(1) = 1$, $T(2) = 4$, $T(3) = 9$, $T(4) = 16$.

   c) Based on the first item and $T(0) = 0$, we have $T(n) = 2 \sum_{i=0}^{n} i - n = 2 \cdot (\frac{n \cdot (n+1)}{2}) - n = n^2$. The same guess/closed-form is obtained from the second item. We verify correctness by substitution: $n^2 = n^2 - 2 \cdot n + 1 + 2 \cdot n - 1 = (n-1)^2 + 2 \cdot n - 1$, and $0^2 = 0$ otherwise.

3) We show $\forall n > 0$, $T(n) \leq 12 \cdot c \cdot n$ by well-founded induction on $n$ ordered by $<$. If $n < 21$, then we are in a base case and we conclude by $T(n) = c \cdot n \leq 12 \cdot c \cdot n$. Otherwise,

$$\begin{aligned} T(n) &= T\left(\left\lceil \frac{2}{10} \cdot n \right\rceil\right) + T\left(\left\lceil \frac{7}{10} \cdot n \right\rceil\right) + c \cdot n \\ &\leq_{\text{IH}} 12 \cdot c \cdot \left\lceil \frac{2}{10} \cdot n \right\rceil + 12 \cdot c \cdot \left\lceil \frac{7}{10} \cdot n \right\rceil + c \cdot n \\ &\leq 12 \cdot c \cdot (1 + \frac{2}{10} \cdot n) + 12 \cdot c \cdot (1 + \frac{7}{10} \cdot n) + c \cdot n \\ &= (12 + \frac{24}{10} \cdot n + 12 + \frac{84}{10} \cdot n + 1) \cdot c \\ &\leq 12 \cdot c \cdot n \end{aligned}$$

   where the last inequality uses that for $n \geq 21$, it holds $25 + \frac{108}{10} \cdot n \leq 12 \cdot n$ as one easily verifies by computation (substituting 21 for $n$) and monotonicity.

4*) The Fibonacci numbers are specified by the recurrence $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ if $n \geq 2$. Being a recurrence there *exists* a *unique* function satisfying this specification. Hence to verify that, for $\phi = \frac{1+\sqrt{5}}{2}$, the closed-form $\frac{\phi^n-(1-\phi)^n}{\sqrt{5}}$ solves the recurrence, it suffices to verify that the three equations hold for the closed-form. This we verify by substituting it in each of the three equations of the recurrence:

a) the equation for $f_0$ is verified by $\frac{\phi^0-(1-\phi)^0}{\sqrt{5}} = \frac{1-1}{\cdots} = 0$;

b) the equation for $f_1$ is verified by $\frac{\phi^1-(1-\phi)^1}{\sqrt{5}} = \frac{2\cdot\phi-1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$.

c) the equation for $f_n$ in case $n \geq 2$, is verified by $\frac{\phi^n-(1-\phi)^n}{\sqrt{5}} = \frac{\phi^{n-1}-(1-\phi)^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2}-(1-\phi)^{n-2}}{\sqrt{5}}$ which follows, by dividing by $\sqrt{5}$ and splitting off factors $\phi^{n-2}$ and $(1-\phi)^{n-2}$, from:

$$\phi^2\cdot\phi^{n-2}-(1-\phi)^2\cdot(1-\phi)^{n-2} = \phi^1\cdot\phi^{n-2}-(1-\phi)^1\cdot(1-\phi)^{n-2}+\phi^0\cdot\phi^{n-2}-(1-\phi)^0\cdot(1-\phi)^{n-2}$$

which in turn follows, by considering the factors of $\phi^{n-2}$ and $(1-\phi)^{n-2}$ separately

$$\begin{aligned} \phi^2 &= \frac{1+2\cdot\sqrt{5}+5}{4} = \frac{1+\sqrt{5}}{2}+1 = \phi^1+\phi^0 \text{ , respectively} \\ (1-\phi)^2 &= \frac{1-2\cdot\sqrt{5}+5}{4} = \frac{1-\sqrt{5}}{2}+1 = (1-\phi)^1+(1-\phi)^0 \end{aligned}$$

Only substitution, no induction, was used in these verifications.

5∗) Computing the first few values and their first and second differences yields: $T(0) = 1$, $T(1) = 0$, $T(2) = 5$, $T(3) = 22$, $T(4) = 57$, $T(5) = 116$, $T(6) = 205$, respectively $U(0) = -1$, $U(1) = 5$, $U(2) = 17$, $U(3) = 35$, $U(4) = 59$, $U(5) = 89$, and $V(0) = 6$, $V(1) = 12$, $V(2) = 18$, $V(3) = 24$, $V(4) = 30$. From the latter having constant difference (of 6) and the hint, we find $T$ is a polynomial of degree 3, so $T(n) = a \cdot n^3 + b \cdot n^2 + c \cdot n + d$ for some $a, b, c, d$. Substituting $0, \ldots, 3$ for $n$ in $T$ yields $d = 1$, $a+b+c+d = 0$, $8\cdot a+4\cdot b+2\cdot c+d = 5$, and $27\cdot a+9\cdot b+3\cdot c+d = 22$. Solving these first yields $d = 1$, and then substituting $c = -1-a-b$ in the last two equations yields $b = 0$, from which it then easily follows that $a = 1$ and $c = -2$, so a closed-form for $T$ is $n^3 - 2\cdot n + 1$.