-) a) By mathematical induction we show $\forall n. \ldots$

- base case: $\sum_{i=1}^{0} 2^i = 0 = 2^1 - 1$; otherwise
- $\sum_{i=1}^{n} 2^i = (\sum_{i=1}^{n-1} 2^i) + 2^n =_{IH} (2^n - 1) + 2^n = 2^{n+1} - 1$, for IH $\sum_{i=1}^{n-1} 2^i = 2^{n+1-1} - 1$.

b) By well-founded induction on $\{(A, \{a\} \cup A) \mid A$ a finite set $, a \notin A\}$ we show $\forall A. \ldots$

- base case: the empty set $\emptyset$ has one subset (itself) and $1 = 2^0 = 2^{|\emptyset|}$.
- we can write a finite set A as $\{a\} \cup A'$ for some finite set A' and $a \notin A'$. The number of subsets of $\{a\} \cup A$ is the twice that of A' since we can adjoin a to each of the latter. From this we conclude since by the IH the number of subsets of A' is $2^{|A'|}$, and $2 \cdot 2^{|A'|} = 2^{1+|A'|} = 2^{|A|}$.

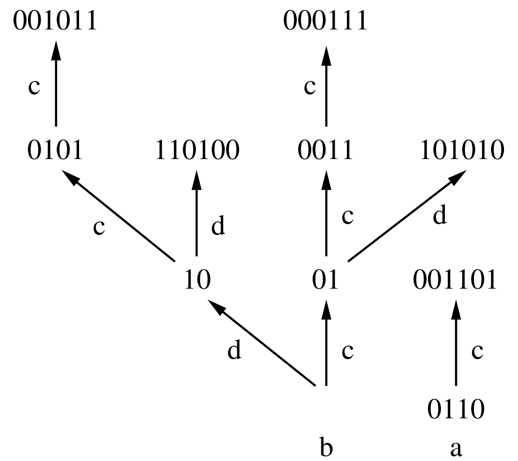
c) By mathematical induction we show $\forall n \ldots.$

- $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!} = \frac{(n-1)!}{k!((n-1)-k)!} + \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} =_{IH} \binom{n-1}{k} + \binom{n-1}{k-1}$, if $0 < k < n$, applying the IH twice for $n-1$.
- otherwise $k = 0$ or $n = k$ and $\frac{n!}{0!n!} = \frac{n!}{n!0!} = 1$.

d) By well-founded induction on $\{(l, l') \mid \ell(l) < \ell(l')\}$ (comparing length) we show $\forall l \ldots.$

- base case: then $\ell(l) = 0$, then x does not occur in l and `bs x []` = `False`.
- if $\ell(l) = 1$, $l =$`[ y ]`. If x occurs in l, `bs x [y]` = `x == y` = `True`, otherwise `False`.
- otherwise $\ell(l) > 1$, and $0 < h = \lfloor \ell(l)/2 \rfloor < \ell(l)$, and accordingly we split $l = [l_0, \ldots, l_{\ell(l)-1}]$ into shorter lists $lt = [l_0, \ldots, l_{h-1}]$ and $ld = [l_h, \ldots l_{\ell(l)-1}]$, both of which are $\leq$-sorted again. if $x < lt$ then by $\leq$-sortedness of l, x occurs in l iff it occurs in lt (as x does *not* occur in ld), and we conclude by the IH for lt as then `bs x l` = `bs x (take h l)` = `bs x` lt. Otherwise, x occurs in l iff it occurs in ld and we conclude by the IH for ld as then `bs x l` = `bs x (drop h l)` = `bs x` ld.

1) - See the figure below, where the rule applied to obtain an element is indicated by labels a–d (on the arrow(s)) below them.



- The well-founded induction principle for the relation $R_W = \{(w, 0w1), (w, 1ww0) \mid w \in W\}$ on W is:
  If ($P(0110)$, and $P(\epsilon)$, and if for all $w \in W$ if $P(w)$ then $P(0w1)$, and if for all $w \in W$ if $P(w)$ then $P(1ww0)$), then for all $w \in W$, $P(w)$.
- We check the four cases, two base cases and two step cases, corresponding to the four clauses, as in the induction principle in the previous item, for the property $P(w) = w$ has the same number of 0s and 1s.
  - $P(0110)$ holds since there are two 0s and two 1s in 0110;
  - $P(\epsilon)$ holds since there are zero 0 and zero 1s in ϵ;
  - Suppose $w \in W$ has the same number n of 0s and 1s (the IH). Then so does $0w1$, namely $n + 1$ of each.
  - Suppose $w \in W$ has the same number n of 0s and 1s (the IH). Then so does $1ww0$, namely $2n + 1$ of each.

2) We make a random instructive selection.
   a) Let f be the relation $\{(1, 0), (0, 0)\}$ on the set of bits $\{0, 1\}$. It is a *function* since both inputs 0, 1 have a unique output, 0. However, its *inverse* $f^{-1} = \{(0, 1), (0, 0)\}$ is not a function on bits, even for two reasons: there is *no* output for the input 1, and the input 0 has *two* outputs 1 and 0;
   b) Let $f = \{(0, 0), (1, 1)\}$ and $g = \{(0, 1), (1, 0)\}$. Both are functions on bits, but their intersection $f \cap g = \emptyset$ is not, as there is *no* output for 0 (nor for 1);
   c) The relations $\leq\, = \{(0, 0), (0, 1), (1, 1)\}$ and $\sqsubseteq\, = \{(0, 0), (1, 0), (1, 1)\}$ are partial orders on bits, but their union $\leq \cup \sqsubseteq\, = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is not since anti-symmetry fails: $(0, 1)$ and $(1, 0)$ are in $\leq \cup \sqsubseteq$ but $0 \neq 1$;
   d) The composition of the above partial orders yields $\leq ; \sqsubseteq\, = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, i.e. the same relation as the intersection which we already know not to be anti-symmetric.
   e) The union $R \cup S = \{(0, 1), (1, 0)\}$ of the well-founded relations $R = \{(0, 1)\}$ and $S = \{(1, 0)\}$ is not well-founded, since we have the infinite descending chain $\ldots 0\ (R \cup S)\ 1\ (R \cup S)\ 0$.

4) A solution is to take the function f that maps $n \in \mathbb{N}$ to $-\frac{n}{2}$ if n is even and to $\frac{n+1}{2}$ if n is odd. For instance $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto -1$, $3 \mapsto 2$, ... The inverse function defined to map an integer $x \in \mathbb{Z}$ to $-2x$ if x is not positive, and to $2x - 1$ otherwise, is easily checked to be the inverse f^{-1} of f by cases, e.g. in case $n \in \mathbb{N}$ is odd, then $f^{-1}(f(n)) = 2(\frac{n+1}{2}) - 1 = n$ as desired, and in case $x \in Z$ is not positive, then $f(f^{-1}(x)) = -\frac{-2x}{2} = x$. Defining for $x, y \in Z$, $x \sqsubseteq y$ if $f^{-1}(x) \leq f^{-1}(y)$ works, for f as above, or in fact for any bijection f between $\mathbb{N}$ and $\mathbb{Z}$, since all properties of $\leq$ carry over: E.g. to check anti-symmetry of $\sqsubseteq$, suppose $x \sqsubseteq y$ and $y \sqsubseteq x$, then $f^{-1}(x) \leq f^{-1}(y)$ and $f^{-1}(y) \leq f^{-1}(x)$, so by anti-symmetry of $\leq$, we have $f^{-1}(x) = f^{-1}(y)$, so also $f(f^{-1}(x)) = f(f^{-1}(y))$ hence $x = y$ by f being a bijective function.