1) - The functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ defined by $f(x) = x+1$, and $g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ defined by $g(x) = x$, are clearly injections. Therefore, by the theorem of Schröder–Bernstein, there exists a bijection between $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$.[1] Note that there are infinitely (uncountably) many possible solutions, e.g. $f(x) = x + \epsilon$ works for any $\epsilon > 0$.
   - The functions $f : (0,1) \to \mathbb{R}_{>0}$ defined by $f(x) = x$, and $g : \mathbb{R} \to (0,1)$ defined by $g(x) = \frac{1}{1+x}$, are clearly injections. Therefore, by the theorem of Schröder–Bernstein, there exists a bijection between $(0,1)$ and $\mathbb{R}_{>0}$.[2]
   - Since we know $|\mathbb{N}| < |\mathbb{R}|$ and trivially $|\mathbb{R}| \leq |\mathbb{R} \times \mathbb{R}|$, e.g. $x \mapsto (x,x)$ is an injection, we have by $\leq$ being a partial order, that $|\mathbb{N}| < |\mathbb{R} \times \mathbb{R}|$ using transitivity.

2) - Every set B_n is non-empty since for all n, $a^n \in B_n$. $B_n \cap B_m = \emptyset$ if $n \neq m$ since ℓ is a function on words. The corresponding equivalence relation is defined by $w =_\ell v$ if $\ell(w) = \ell(v)$. The cardinality of $B_n = 2^n$.
   - Reflexivity and symmetry are trivial consequences of reflexivity of equality on natural numbers and commutativity of multiplication, respectively. If for fractions $\frac{n}{m}$, $\frac{k}{\ell}$, $\frac{p}{q}$, we have $n \cdot \ell = m \cdot k$ and $k \cdot q = \ell \cdot p$, then $n \cdot q \cdot k = n \cdot \ell \cdot p = m \cdot k \cdot p$, hence by cancelling k on both sides, $n \cdot q = m \cdot p$. A system of representatives is $\{\frac{n}{m} \mid \gcd(n,m) = 1\}$, the set of *normalised* fractions.

3) - By subtraction: $\gcd(42,63) = \gcd(42,21) = \gcd(21,21)$ so the common integer 21 is the gcd. By division: $\gcd(42,63) = \gcd(42,21)$ so the divisor 21 is the gcd, as $21 \mid 42$.
   -

$$
\begin{array}{lrrrl}
(1) & 77 = & 1 \cdot 77 + & 0 \cdot 30 & \\
(2) & 30 = & 0 \cdot 77 + & 1 \cdot 30 & \\
(3) & 47 = & 1 \cdot 77 + & (-1) \cdot 30 & (1) - (2) \\
(4) & 17 = & 1 \cdot 77 + & (-2) \cdot 30 & (3) - (2) \\
(5) & 13 = & (-1) \cdot 77 + & 3 \cdot 30 & (2) - (4) \\
(6) & 4 = & 2 \cdot 77 + & (-5) \cdot 30 & (4) - (5) \\
(7) & 9 = & (-3) \cdot 77 + & 8 \cdot 30 & (5) - (6) \\
(8) & 5 = & (-5) \cdot 77 + & 13 \cdot 30 & (7) - (6) \\
(9) & 1 = & (-7) \cdot 77 + & 18 \cdot 30 & (8) - (6)
\end{array}
$$

   Indeed $(-7) \cdot 77 + 18 \cdot 30 = -539 + 540 = 1$.

4∗) The bijection f' between $\mathbb{N}$ and $\mathbb{Z}$ constructed from f, g according to the proof of the theorem of Schröder–Bernstein on slide 13 of week 8, maps x to $g^{-1}(x) = n$ if $x = 2^n$ and $n \in N$ is not itself a power of 2 and $x \neq 0$, and to $g^{-1}(x) = -n$ if $x = 3^n$ for $n \in N$ and positive

[1] The bijection f' constructed from f, g according to the proof of the theorem, on slide 13 of week 8, maps x to $g^{-1}(x) = x$ if $x \notin \mathbb{N}$ (x-chain ends in interval $(0,1)$ on the right), and to $x+1$ if $x \in \mathbb{N}$ (x-chain ends in $\{1\}$ on the left).

[2] The bijection f' constructed from f, g according to the proof of the theorem, on slide 13 of week 8, maps x to $g^{-1}(x) = \frac{1}{x} - 1$ if $x \neq \frac{-1+\sqrt{5}}{2}$ (x-chain ends in a number >1 on the right; applying g^{-1} to a value $0 < x < 1$ increases the *distance* to $\frac{-1+\sqrt{5}}{2}$ (the golden ratio minus 1)) and to $f(x) = x = \frac{-1+\sqrt{5}}{2}$ otherwise (x-chain is ∞). Thus, $f = g^{-1}$, i.e. g is already a bijection.

(x-chain ends on the right in both cases), and to $f(x) = x$ otherwise (either x-chain ends on the left, or it is ∞ if x is 0 or of shape 2^{2^n} for some $n \in N$). In particular $f'(0) = f(0) = 0$, $f'(1) = f(1) = 1$, $f'(2) = f(2) = 2$, $f'(3) = g^{-1}(3) = -1$, $f'(4) = f(4) = 4$ since $4 = 2^{2^1}$ and $f'(n) = f(n) = n$ if $5 \leq n \leq 7$ since these are not powers of 2 or 3, $f'(8) = g^{-1}(8) = 3$ as $8 = 2^3$ and 3 is not a power of 2, $f'(9) = g^{-1}(9) = -2$ as $9 = 2^3$, etc.

5$*$) By exactly the same proof as on slide 12 of lecture 7, replacing (cardinalities of) intersections $\bigcap$ and unions $\bigcup$ with gcd and lcm respectively, minus $-$ with division $\div$ and sum $\sum$ with product $\prod$.