Summary last week

- divide and conquer algorithms, e.g. mergesort
- have asymptotic complexities given by recurrences $T(n) = \ldots T(< n) \ldots$
- may find a closed-form solution for a recurrence by:
- self-substitution and looking for pattern; or
- guessing and verifying; or
- generating functions (not this course); or
- master theorem: $T(n) = a \cdot T(\frac{n}{b}) + f(n)$ if $n = b^k$ for $k > 0$, otherwise c:

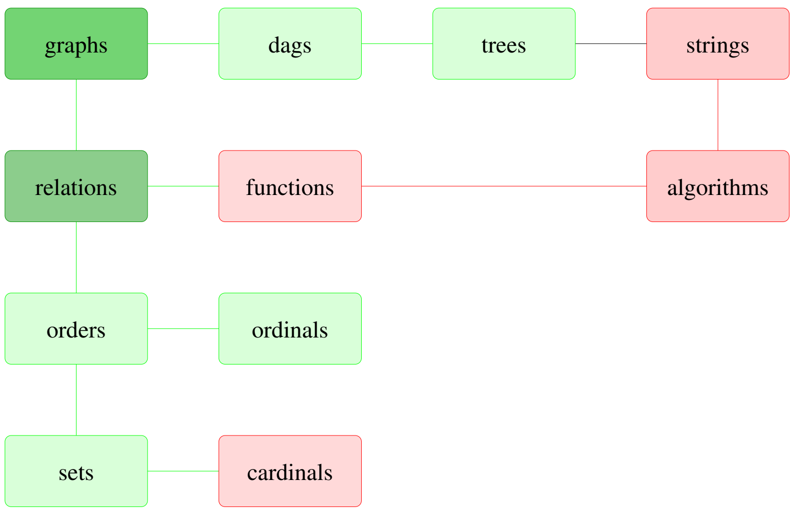
$$T(n) \in \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b^s \\ \Theta(n^s \log n) & \text{if } a = b^s \\ \Theta(n^s) & \text{if } a < b^s \end{cases}$$

for T increasing, $a \geqslant 1$, $b > 1$, $c > 0$, and $f \in \Theta(n^s)$ with $s \geqslant 0$.

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Limitations of algorithms (recall from 3rd lecture)

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Remark

These limitations will be addressed in the last few weeks of course (i.e. now)

Function defined by a TM (recall from 3rd lecture)

Definition

a TM M

- accepts $x \in \Sigma^*$, if $\exists\, y, n$:

$$(s, \vdash x \sqcup^\infty, 0) \xrightarrow[M]{*} (t, y, n)$$

- rejects $x \in \Sigma^*$, if $\exists\, y, n$:

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- halt on input x, if x is accepted or rejected
- does not halt on input x, if x is neither accepted nor rejected
- is total, if M halts on all inputs

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Definition

A function $f : A \to B$ is defined by a TM M for every $x \in A$, M accepts input x with $f(y)$ on the tape (and does not halt or rejects on inputs $x \notin A$).

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Idea of computability

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Definition (computability via TM)

$f : \mathbb{N} \to \mathbb{N}$ computable if it can be defined by a TM

Examples of computable functions

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- functions defined by finite cases
  $f(n) = n$ if n odd, otherwise n^2

Limits of computability

Lemma

there exist functions that are not computable (more functions than programs)

Proof.

Limits of computability

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Theorem

concrete non-computable functions (diagonalise away from TM behaviours)

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Theorem

concrete non-computable functions

rest of this lecture, details of the above: coding, diagonalising way

Recursive/recursively enumerable languages

Definition

A language L (or, more generally, a set) is

- recursively enumerable, if there exists a TM M such that $L = \mathrm{L}(M)$
  i.e. L is the set of strings accepted by M
- recursive, if there exists a total TM M, such that $L = \mathrm{L}(M)$
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Computable function vs. recursive sets

Partial function $f : \mathbb{N} \to \mathbb{N}$ is computable iff $L_f = \{x\#f(x) \mid x \in \mathbb{N}\}$ is recursively enumerable. Total f is computable iff L_f is recursive.

Theorem

Let $L \subseteq \Sigma^*$ be a recursive language over some alphabet Σ; then $\sim L$ is recursive.

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Proof.

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Proof.

The first part of the theorem follows from the definitions; the second part we will show later ∎

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- define TM M', such that its tape has two 'halves' (or a TM with 2-tapes):

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Decidable/semi-decidable properties

Definition

Let Σ be an alphabet. A property P of words over Σ is

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Every decidable problem is semi-decidable

Remark

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Encoding TMs

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1. Number of states
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3. input and tape alphabet
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represents $Q = \{0, \ldots, n-1\}$, $\Gamma = \{0, \ldots, m-1\}$, $\Sigma = \{0, \ldots, k-1\}$, $(k \leqslant m)$, s initial state, t accepting state, r rejecting state, u left-end marker, v blank symbol

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Example (Continued)

consider M and encode $\delta(p, a) = (q, b, d)$, where $c = 0$ if $d = \mathrm{L}$ and $c = 1$ if $d = \mathrm{R}$

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We encode $M' = (\{s, p, t, r\}, \{0, 1\}, \{0, 1, \vdash, \sqcup\}, \vdash, \sqcup, \delta, s, t, r)$ by

	$\vdash$	0	1	$\sqcup$
s	$(s, \vdash, \mathrm{R})$	$(s, 0, \mathrm{R})$	$(s, 1, \mathrm{R})$	$(p, \sqcup, \mathrm{L})$
p	$(t, \vdash, \mathrm{R})$	$(t, 1, \mathrm{L})$	$(p, 0, \mathrm{L})$	$\cdot$

We obtain

$$\underbrace{0000}_{n=4}1\underbrace{0000}_{m=4}1\underbrace{00}_{k=2}1\underbrace{\epsilon}_{s}1\underbrace{00}_{t}1\underbrace{000}_{r}1\underbrace{00}_{\vdash}1\underbrace{000}_{\sqcup}1\cdots$$

and, for example, $\delta(p, \vdash) = (t, \vdash, \mathrm{R})$ yields $010^{2}10^{2}10^{2}101$

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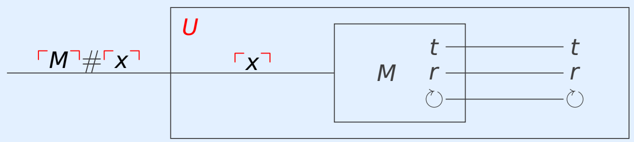
UTM schematically



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Remark

Meta-programming and macros originate with UTMs

Definition

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Enumerating all Turing machines

$$M_\epsilon, M_0, M_1, M_{00}, M_{01}, M_{10}, M_{11}, M_{000}, \dots$$

(ordered with respect to the lexical order)

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	ϵ	0	1	00	01	10	11	000	001	010	...
M_ϵ	!	↻	↻	!	!	↻	!	↻	!	!	
M_0	↻	↻	!	!	↻	!	!	↻	↻	!	
M_1	↻	!	↻	!	↻	!	!	↻	↻	!	
M_{00}	!	↻	↻	!	!	!	!	↻	↻	!	
M_{01}	!	!	!	!	↻	↻	↻	!	!	↻	...
M_{10}	!	!	↻	!	!	↻	!	!	↻	!	
M_{11}	!	!	↻	↻	!	↻	!	↻	!	↻	
M_{000}	!	!	!	!	↻	!	!	↻	!	↻	
M_{001}	↻	!	!	!	!	↻	!	!	!	!	
$\vdots$						$\vdots$					$\ddots$

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Given an enumeration m_ϵ, m_0, m_1, ... of such behaviours, indexed by finite bit-strings

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$$cd(x) = \begin{cases} ↻ & \text{if } m_x(x) = ! \\ ! & \text{if } m_x(x) = ↻ \end{cases}$$

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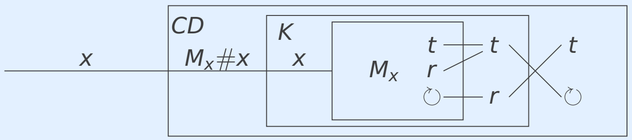

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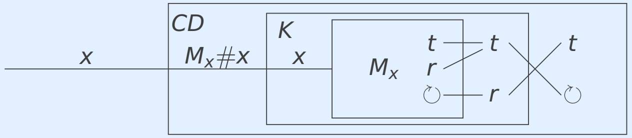

3. CD exhibits behaviour cd. For the earlier behaviour matrix we, e.g., have:
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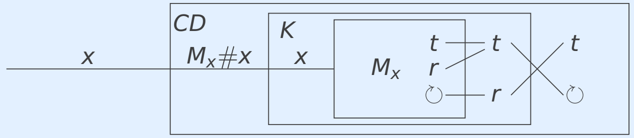

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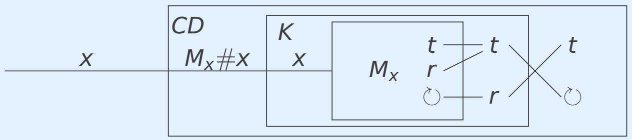

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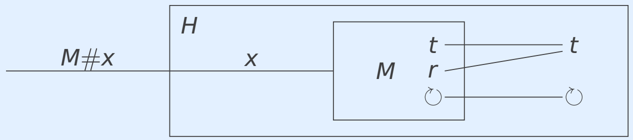

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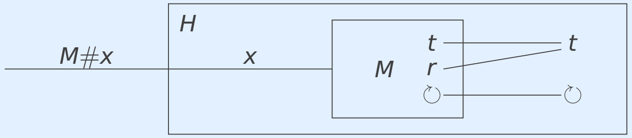

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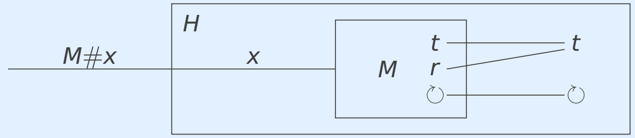

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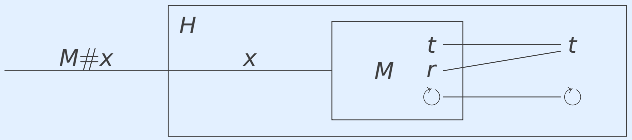

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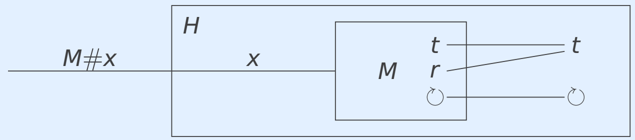

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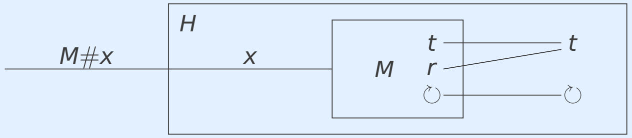

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