Summary last week

- function $f : \mathbb{N} \to \mathbb{N}$ computable if exists effective procedure computing $f(x)$ on x
- effective procedure for f if exists TM M that leaves output $f(x)$ on tape on input x;
- equivalently defined via other models of computation: μ-recursion, λ-calculus,...
- language L recursive(ly enumerable) if exists (total) TM M accepting L ($L = L(M)$)
- property P (semi-)decidable if $\{x \mid P(x)\}$ is recursive(ly enumerable)

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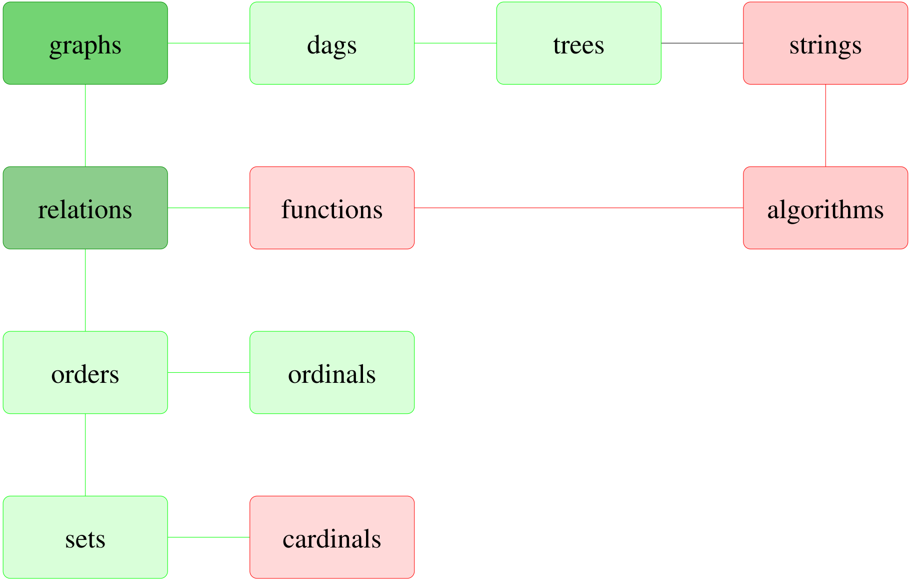
- language L recursive(ly enumerable) if exists (total) TM M accepting L ($L = L(M)$)
- property P (semi-)decidable if $\{x \mid P(x)\}$ is recursive(ly enumerable)
- TM encoded by some $x \in \{0,1\}^*$ (program as bit-string) $\Rightarrow$ countably many TMs
- uncountably many functions $\mathbb{N} \to \mathbb{N}$ $\Rightarrow$ some (most) functions not computable
- exist TM U that is universal: U on $x\#y$ simulates TM M_x (TM having code x) on y
- diagonal d is behaviour exhibited when running M_x on x (itself) for each input x
- complement cd of d distinct from all TM behaviours $\Rightarrow$ not a TM behaviour
- halting problem HP $:= \{M\#x \mid M \text{ halts for input } x\}$ not recursive
- if L and $\sim L$ recursively enumerable, then (both) recursive

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Recursive/recursively enumerable languages

Definition

A language L (or, more generally, a set) is

- recursively enumerable, if there exists a TM M such that $L = \mathrm{L}(M)$
  i.e. L is the set of strings accepted by M
- recursive, if there exists a total TM M, such that $L = \mathrm{L}(M)$
  i.e. M is required to halt (accept or reject) on all strings

Theorem

- *every recursive set is recursively enumerable;*
- *if a set and its complement are recursively enumerable, they are recursive;*

A non-recursive language HP (recapitulation)

Definition (Halting Problem)

HP $:= \{M\#x \mid M$ halts on input $x\}$

Theorem

HP *is not recursive*

Definition

behaviour is a map from input words $x \in \{0,1\}^*$ to either ! (halts) or ↻ (loops).

Proof of Theorem.

Suppose HP were recursive, i.e. there is a total TM K such that $L(K) = \text{HP}$.

1. there is a behaviour *cd* not exhibited by any TM;
2. using K we could construct a TM *CD* exhibiting behaviour *cd*.

Contradiction, so HP is not recursive. ∎

(1) there is a behaviour *cd* not exhibited by any TM

Proof.

Enumerating all TMs as $M_\epsilon, M_0, M_1, M_{00}, \ldots,$ their behaviours can be depicted as:

	ϵ	0	1	00	01	10	11	000	001	010	$\ldots$
M_ϵ	!	$\circlearrowright$	$\circlearrowright$	!	!	$\circlearrowright$	!	$\circlearrowright$	!	!	
M_0	$\circlearrowright$	$\circlearrowright$	!	!	$\circlearrowright$	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_1	$\circlearrowright$	!	$\circlearrowright$	!	$\circlearrowright$	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{00}	!	$\circlearrowright$	$\circlearrowright$	!	!	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{01}	!	!	!	!	$\circlearrowright$	$\circlearrowright$	$\circlearrowright$	!	!	$\circlearrowright$	$\ldots$
M_{10}	!	!	$\circlearrowright$	!	!	$\circlearrowright$	!	!	$\circlearrowright$	!	
$\vdots$						$\vdots$					$\ddots$

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M_1	$\circlearrowright$	!	$\circlearrowright$	!	$\circlearrowright$	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{00}	!	$\circlearrowright$	$\circlearrowright$	!	!	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{01}	!	!	!	!	$\circlearrowright$	$\circlearrowright$	$\circlearrowright$	!	!	$\circlearrowright$	...
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diagonal behaviour $d = !\ \circlearrowright\ \circlearrowright\ !\ \circlearrowright\ \circlearrowright\ \ldots$ ■

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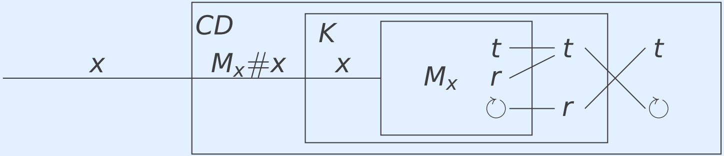
M_1	$\circlearrowright$	!	!	!	$\circlearrowright$	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{00}	!	$\circlearrowright$	$\circlearrowright$	$\circlearrowright$	!	!	!	$\circlearrowright$	$\circlearrowright$	!	
M_{01}	!	!	!	!	!	$\circlearrowright$	$\circlearrowright$	!	!	$\circlearrowright$	$\ldots$
M_{10}	!	!	$\circlearrowright$	!	!	!	!	!	$\circlearrowright$	!	
$\vdots$						$\vdots$					$\ddots$

complement diagonal $cd = \circlearrowright \; ! \; ! \; \circlearrowright \; ! \; ! \; \ldots$ not exhibited by any TM ∎

(2) using K we could construct a TM CD exhibiting behaviour cd

Proof.

Suppose K were a total TM K such that $L(K) = \text{HP}$. Construct CD:



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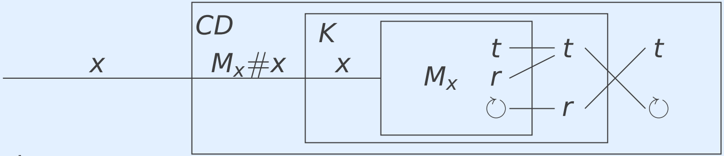

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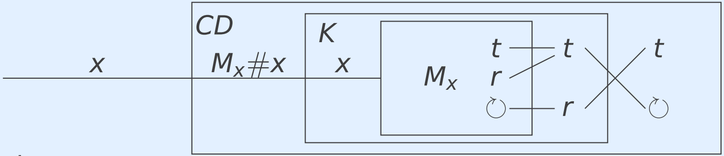

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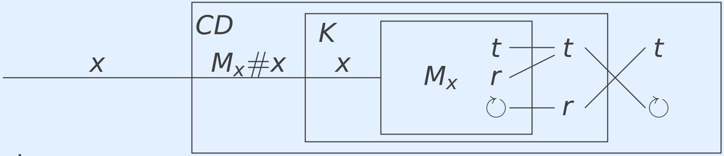

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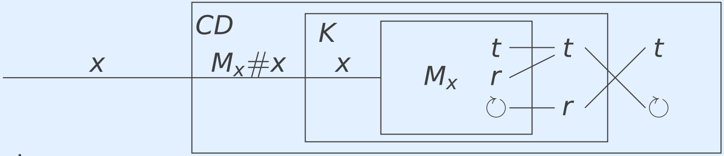

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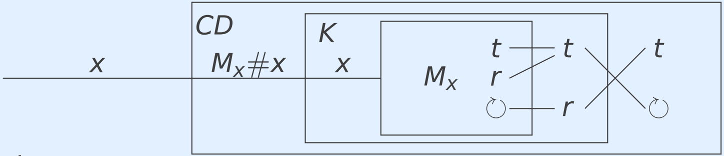

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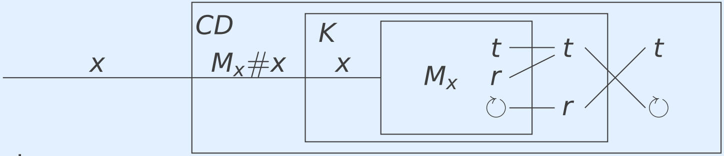

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CD exhibits behaviour cd ∎

Complements

Lemma

if L recursive, then so is $\sim L$

Proof.

If $L = L(M)$ for total TM M, then $\sim L = L(M')$ for M' as M but swapping accept, reject ∎

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Lemma

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Proof.

If $L = L(M)$ for total TM M, then $\sim L = L(M')$ for M' as M but swapping accept, reject ∎

Theorem

$\sim$HP *is not recursively enumerable (although* HP *is)*

Proof.

Suppose $\sim$HP were recursively enumerable.

- then both HP (previous lecture) and $\sim$HP would be recursively enumerable;
- so both HP and $\sim$HP would in fact be recursive (previous lecture);
- but that would contradict that HP is not recursive (previous lecture).

So $\sim$HP is not recursively enumerable. ∎

Closure properties of recursively enumerable languages

Theorem

recursively enumerable languages are closed under union and intersection, but not under complement or difference

Proof.

- if $L_1 = L(M_1)$, $L_2 = L(M_2)$, running both M_1, M_2 on a given input x, and accepting if at least one/both of them accepts, shows closure under union/intersection.

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Corollary

recursive languages are closed under union, intersection, complement, and difference

by above closure properties and using De Morgan; $\sim(L_1 \cup L_2) = (\sim L_1) \cap (\sim L_2)$

Another non-recursive language MP

Definition (Membership Problem)

$\text{MP} := \{M\#x \mid M \text{ accepts input } x\}$

Theorem

MP *is not recursive*

Proof.

Suppose MP were recursive, i.e. there is a total TM K such that $L(K) = \text{MP}$.

1. there is a language *cd* not accepted by any TM;
2. using K we could construct a TM *CD* accepting *cd*.

Contradiction, so MP is not recursive. ∎

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	ϵ	0	1	00	01	10	11	000	001	010	$\ldots$
M_ϵ	×	×	×	✓	✓	×	×	×	×	✓	
M_0	×	×	✓	✓	×	✓	✓	×	×	✓	
M_1	×	×	×	✓	×	×	✓	×	×	✓	
M_{00}	✓	×	×	✓	✓	✓	✓	×	×	✓	
M_{01}	×	✓	×	✓	×	×	×	✓	✓	×	$\cdots$
M_{10}	✓	✓	×	✓	×	×	×	×	×	×	
$\vdots$						$\vdots$					$\ddots$

(1) there is a language $\overline{d}$ not accepted by any TM

Proof.

Enumerating all TMs as $M_\epsilon, M_0, M_1, M_{00}, \ldots$, their languages can be depicted as:

	ϵ	0	1	00	01	10	11	000	001	010	...
M_ϵ	×	×	×	✓	✓	×	×	×	×	✓	
M_0	×	×	✓	✓	×	✓	✓	×	×	✓	
M_1	×	×	×	✓	×	×	✓	×	×	✓	
M_{00}	✓	×	×	✓	✓	✓	✓	×	×	✓	
M_{01}	×	✓	×	✓	×	×	×	✓	✓	×	...
M_{10}	✓	✓	×	✓	×	×	×	×	×	×	
$\vdots$						$\vdots$					$\ddots$

diagonal language $d = \{00, \ldots\}$ ■

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	ϵ	0	1	00	01	10	11	000	001	010	$\ldots$
M_ϵ	$\checkmark$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\checkmark$	
M_0	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$	
M_1	$\times$	$\times$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$	$\times$	$\times$	$\checkmark$	
M_{00}	$\checkmark$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\checkmark$	
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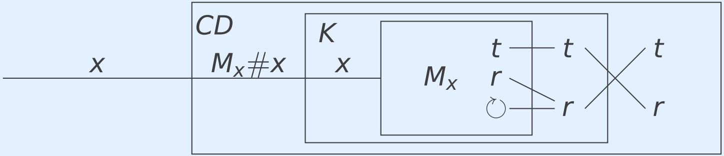
complement diagonal $cd = \{\epsilon, 0, 1, 01, 10, 11, \ldots\}$ not accepted by any TM ∎

(2) using K we could construct a TM CD accepting cd

Proof.

Suppose K were a total TM K such that $L(K) = \mathrm{MP}$. Construct CD:



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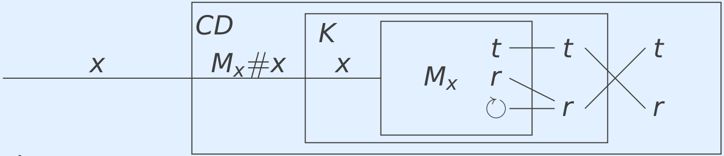

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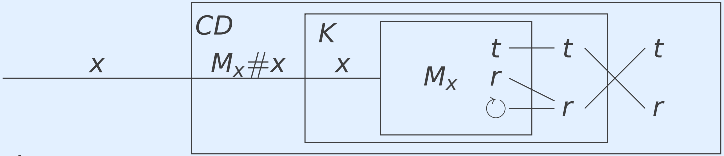

in words:

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(2) using K we could construct a TM CD **accepting** cd

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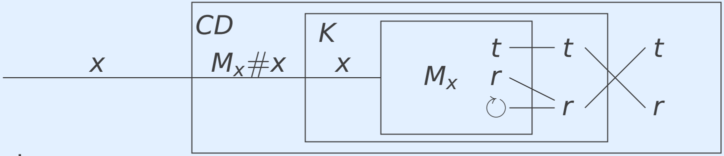

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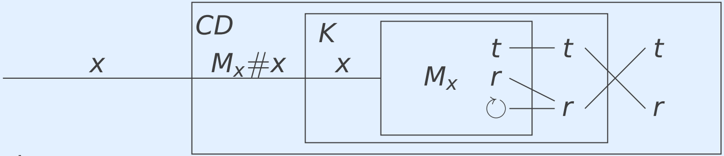

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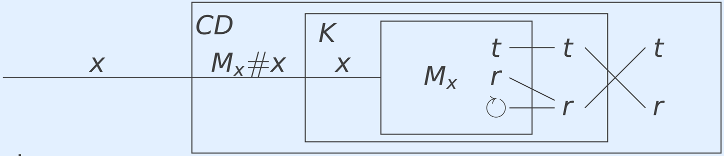

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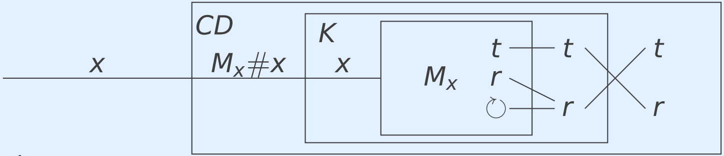

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CD accepts cd ∎

Border between in/outside of language

Remark

- language L being recursively enumerable (recursive) depends on its border
- how 'difficult' it is to decide that string x is in L (and not in L)
- does not depend on cardinality of L
- but note every finite language recursive (finite case-distinction)

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Example

for A set of all strings over alphabet $\supseteq \{0, 1, \#\}$

- $\emptyset \subseteq \mathrm{HP} \subseteq A$

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- although both $\emptyset, A$ recursive, HP is not recursive
- sub/superset of recursive or recursively enumerable language need not be so
- may have $L_0 \subset L_1 \subset L_2 \subset L_3 \subset \ldots$ such that L_{2i} recursive, L_{2i+1} not recursive

Relating non-recursiveness of HP and MP

similarity between proofs

suppose $L = L(K)$ for some total TM K

- determine cd distinct from $L(M)$ for every TM M by diagonalising away
- show that using K we could construct TM CD with $L(CD) = cd$

so supposition must be false

avoid redoing diagonalisation?

Relating non-recursiveness of HP and MP

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- solve problem L using solution for problem L' as sub-routine

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- say: reduce L to L' (reduction from L to L')

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reducing problems to each other

- solve problem L using solution for problem L' as sub-routine
- say: reduce L to L'
- $x \in L$ iff $f(x) \in L'$ using (computable!) function f

Relating non-recursiveness of HP and MP

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reducing problems to each other

- solve problem L using solution for problem L' as sub-routine
- say: reduce L to L'
- $x \in L$ iff $f(x) \in L'$ using function f
- L is not more complex than L' (may be strictly less so; may be simpler subroutine)

Relating non-recursiveness of HP and MP

similarity between proofs

suppose $L = L(K)$ for some total TM K

- determine cd distinct from $L(M)$ for every TM M by diagonalising away
- show that using K we could construct TM CD with $L(CD) = cd$

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- if L not recursive and $L \leq L'$, then L' not recursive

Formal definition of reduction

Definition (recall from before)

$f : \Sigma^* \to \Sigma^*$ is computable if $\exists$ a total TM T with input alphabet Σ, such that on input $x \in \Sigma^*$, T writes $f(x)$ on the tape

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Theorem

if L not recursive and $L \leq L'$, then L' is not recursive

Proof.

supposing $L \leq L'$ by function f by TM T and L' were recursive by total TM K, then $x \in L$ iff $f(x) \in L'$ iff K accepts $f(x)$. that is, L is recursive using TM that first transforms x into $f(x)$ by running T and then runs K on $f(x)$. contradiction. so L' is not recursive. ∎

Reducing HP to MP ?

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Lemma

$M\#x \in \text{HP}$ *iff* $f(M\#x) \in \text{MP}$

Proof.

$M\#x \in \text{HP} \Rightarrow M$ halts on $x \Rightarrow M'$ accepts $x \Rightarrow M'\#x \in \text{MP}$

$M\#x \notin \text{HP} \Rightarrow M$ does not halt on $x \Rightarrow M'$ does not halt on $x \Rightarrow M'\#x \notin \text{MP}$ ∎

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MP *is not recursive*

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since HP is not recursive (assumed known) and $\mathrm{HP} \leq \mathrm{MP}$ ∎

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Lemma

$M\#x \in \mathrm{MP}$ *iff* $f(M\#x) \in \mathrm{HP}$

Proof.

$M\#x \in \mathrm{MP} \;\Rightarrow\; M$ accepts $x \;\Rightarrow\; M'$ accepts $x \;\Rightarrow\; M'\#x \in \mathrm{HP}$

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More on reducibility

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- to show transitivity, compose the functions (run the TMs consecutively) ∎

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Lemma

there is no program testing whether a given program is a "hello world"-program (prints "hello world" and then accepts)

More on reducibility

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Lemma

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Proof.

show HP $\leq$ "hello world"-program. let $f(M\#x)$ be $M'\#x$ with M' a TM that first runs M on its input, if that halts overwrites tape with "hello world" and then accepts:

$M\#x \in \text{HP} \Rightarrow M$ halts on $x \Rightarrow M'$ is a "hello world"-program

$M\#x \notin \text{HP} \Rightarrow M$ does not halt on $x \Rightarrow M'$ is not a "hello world"-program ∎

Regular languages

Question

What languages can be accepted for machines more restricted than TMs?

Regular languages

We consider finite automata. These accept regular languages, and will show these are recursive, but not necessarily the other way around,

relevance of regular languages

- software for designing and testing of digital circuits
- software components of compiler, e.g. for lexical analysis:
- software for searching in long texts
- software to verify all kinds of systems having a finite number of states
- components of computer games (computer-controlled non-player-character)

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Example

$\emptyset$ and the set of all strings are regular, as are all finite languages.

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Definition

A DFA is a 5-tuple $A = (Q, \Sigma, \delta, s, F)$ with

1. Q a finite set of states
2. Σ a finite set of input symbols, (Σ is called the input alphabet)
3. $\delta\colon Q \times \Sigma \to Q$ the transition function
4. $s \in Q$, the start or initial state
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Beware: δ must be defined, for all possible inputs

Transition table

	$a_1 \in \Sigma$	$a_2 \in \Sigma$	$\cdots$
$q_1 \in Q$	$\delta(q_1, a_1)$	$\delta(q_1, a_2)$	$\cdots$
$q_2 \in Q$	$\delta(q_2, a_1)$		
$\vdots$	$\vdots$		

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Transition graph

For a DFA $A = (Q, \Sigma, \delta, s, F)$, its (directed) transition graph with initial state d and final states F where:

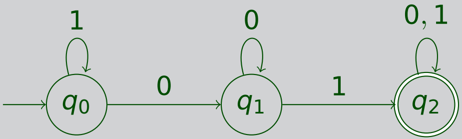
1. the states are the nodes
2. the edges E are
$$(p, q) \qquad p, q \in Q \text{ and } \exists a \in \Sigma \text{ with } \delta(p, a) = q$$
3. the edges are labelled by symbols by a function $b \colon E \to \Sigma$ defined by
$$(p, q) \mapsto a \qquad \text{if } \delta(p, a) = q$$

Example

The DFA $A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$ with transition table

	0	1
$\rightarrow q_0$	q_1	q_0
q_1	q_1	q_2
$*q_2$	q_2	q_2

has the following transition graph:



Definition (extending the transition function)

Let δ be a transition function. The extended transition function $\hat{\delta}\colon Q \times \Sigma^* \to Q$ is inductively defined by:

$$\begin{aligned} \hat{\delta}(q, \epsilon) &:= q \\ \hat{\delta}(q, xa) &:= \delta(\hat{\delta}(q, x), a) & \qquad x \in \Sigma^*,\ a \in \Sigma \end{aligned}$$

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Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA; the language $\mathsf{L}(A)$ accepted by A is:

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For the DFA A above, $\hat{\delta}(q_0, 0010) = q_2$

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For the DFA A, we have $\mathrm{L}(A) = \{x01y \mid x, y \in \Sigma^*\}$. The language $\mathrm{L}(A)$ is the set of all words in which 01 occurs somewhere (or rather of words not of the form: a number of 1s followed by a number of 0s)

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Definition

A formal language L is regular, if $\exists$ DFA A, such that $\mathrm{L}(A) = L$