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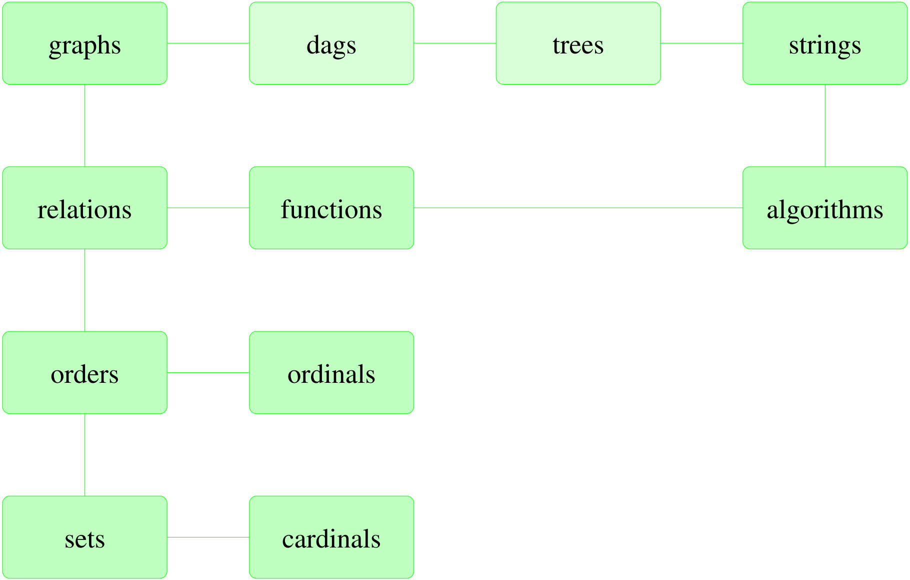
- recursive languages closed under union, intersection, complement, difference
- f is reduction from L to L' if f computable and $\forall x$, $x \in L$ iff $f(x) \in L'$
- $L \leq L'$, L reducible to L', if there exists reduction f from L to L'
- if L non-recursive and $L \leq L'$ then L' is non-recursive
- MP $\leq$ HP and HP $\leq$ MP

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Regular languages

Question

What languages can be accepted for machines more restricted than TMs?

Regular languages

We consider finite automata. These accept regular languages, and will show these are recursive, but not necessarily the other way around,

relevance of regular languages

- software for designing and testing of digital circuits
- software components of compiler, e.g. for lexical analysis:
- software for searching in long texts
- software to verify all kinds of systems having a finite number of states
- components of computer games (computer-controlled non-player-character)

Deterministic finite automata (DFAs)

Example

$\emptyset$ and the set of all strings are regular, as are all finite languages.

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Definition

A DFA is a 5-tuple $A = (Q, \Sigma, \delta, s, F)$ with

1. Q a finite set of states
2. Σ a finite set of input symbols, (Σ is called the input alphabet)
3. $\delta \colon Q \times \Sigma \to Q$ the transition function
4. $s \in Q$, the start or initial state
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Beware: δ must be defined, for all possible inputs

Transition table

	$a_1 \in \Sigma$	$a_2 \in \Sigma$	$\cdots$
$q_1 \in Q$	$\delta(q_1, a_1)$	$\delta(q_1, a_2)$	$\cdots$
$q_2 \in Q$	$\delta(q_2, a_1)$		
$\vdots$	$\vdots$		

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Transition graph

For a DFA $A = (Q, \Sigma, \delta, s, F)$, its (directed) transition graph with initial state d and final states F where:

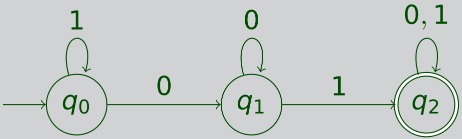
1. the states are the nodes
2. the edges E are
$$(p, q) \qquad p, q \in Q \text{ and } \exists a \in \Sigma \text{ with } \delta(p, a) = q$$
3. the edges are labelled by symbols by a function $b \colon E \to \Sigma$ defined by
$$(p, q) \mapsto a \qquad \text{if } \delta(p, a) = q$$

Example

The DFA $A = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, q_0, \{q_2\})$ with transition table

	0	1
$\rightarrow q_0$	q_1	q_0
q_1	q_1	q_2
$*q_2$	q_2	q_2

has the following transition graph:



Definition (extending the transition function)

Let δ be a transition function. The extended transition function $\hat{\delta}\colon Q \times \Sigma^* \to Q$ is inductively defined by:

$$\begin{aligned} \hat{\delta}(q, \epsilon) &:= q \\ \hat{\delta}(q, xa) &:= \delta(\hat{\delta}(q, x), a) \qquad\qquad x \in \Sigma^*,\ a \in \Sigma \end{aligned}$$

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Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA; the language $\mathsf{L}(A)$ accepted by A is:

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For the DFA A above, $\hat{\delta}(q_0, 0010) = q_2$

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For the DFA A, we have $L(A) = \{x01y \mid x, y \in \Sigma^*\}$. The language $L(A)$ is the set of all words in which 01 occurs somewhere (or rather of words not of the form: a number of 1s followed by a number of 0s)

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Definition

A formal language L is regular, if $\exists$ DFA A, such that $\mathrm{L}(A) = L$

Closedness of the regular languages

Theorem

] Let L, M be regular languages (over the alphabet Σ). Then

1. *the complement $\sim L$ is regular*
2. *the intersection $L \cap M$ is regular*
3. *the union $L \cup M$ ist regular*
4. *the set difference $L \setminus M$ ist regular*

Sketch.

- swap accept/not-accept states
- pair of states; (q, q') accept if q and q' accept
- pair of states: (q, q') accept if q or q' accept
- $L \setminus M = L \cap \sim M$ and previous items

Limitations of finite automata

Example

Consider the language

$$B = \{a^n b^n \mid n \geqslant 0\} = \{\epsilon, ab, aabb, aaabbb, \ldots\}$$

The language B is not regular (note that it is recursive)

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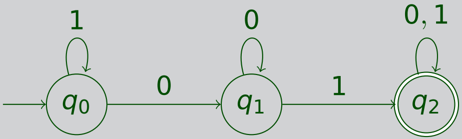
Example

Consider the language

$$C = \{0^{2^n} \mid n \geqslant 0\} = \{0, 00, 0000, 00000000, \ldots\}$$

The language C is not regular

Example



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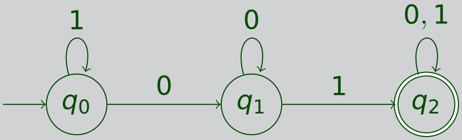

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What can we say about the states the automaton goes ‘through’ to accept the word $w = 0000110$?

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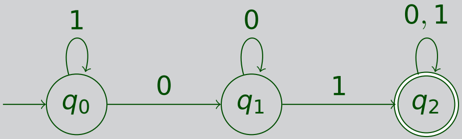

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What can we say about the states the automaton goes ‘through’ to accept the word $w = 0000110$?

Answer

since $\ell(w) = 7 > 3 = |Q|$ the automaton must go through some state at least twice; the automaton cycles

Theorem (Pumping lemma)

Let L be a regular language over Σ. Then there exists a number $n \in \mathbb{N}$, such that for all words $w \in L$ of length at least n ($\ell(w) \geq n$), there exist words $x, y, z \in \Sigma^*$ such that $w = xyz$ and

- $y \neq \epsilon$;
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$$w = w_1 \cdots w_m \in L$$

with $w_1, \ldots, w_m \in \Sigma$ and $m \geq n$

Proof. (continued).

- define $p_l := \hat{\delta}(s, w_1 \cdots w_l)$;
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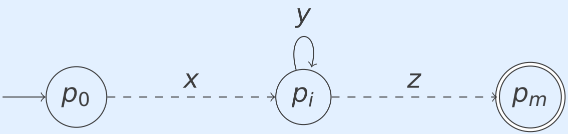

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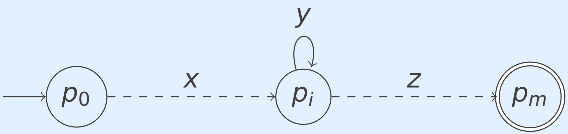

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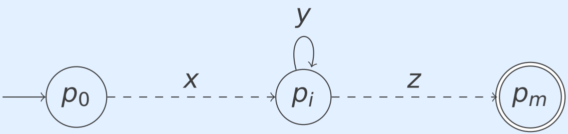

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Application of the pumping lemma

Theorem (Application (1))

Let L be a formal language over Σ such that:

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Example (1)

Let $\Sigma = \{1\}$; then

$$D = \{w \in \Sigma^* \mid \ell(w) \text{is a prime number}\}$$

not regular

Example (2)

We show that for D we have

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Then $v \notin L$, since

$$\ell(v) = \ell(x(y)^{(p-m)}z) = (p - m) + m \cdot (p - m) = (p - m) \cdot (m + 1)\,.$$

That is, $\ell(v)$ is not a prime number, if $(p - m) > 1$ and $(m + 1) > 1$

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we have $x(y)^0 z \notin E$, so the conditions of the pumping lemma are satisfied, hence L is not regular