Summary last week

- correctness and complexity statements and proofs of Floyd's algorithm
- proof method: proof by contradiction
- relations as digraphs; whether elements relate, not how
- properties of relations: reflexivity, symmetry, transitivity, . . .
- closing a relation with respect to a property
- Warshall's transitive closure algorithm
- functions as relations; every element related to some unique element
- defining functions by specifications

Summary last week

Theorem

The following overwrites the adjacency matrix A of R, with that of R^+

For r from 0 to $n-1$ repeat:
  Set $N = A$.
  For i from 0 to $n-1$ repeat:
    For j from 0 to $n-1$ repeat:
      Set $N_{ij} = max(A_{ij}, A_{ir} \cdot A_{rj})$
  Set $A = N$.

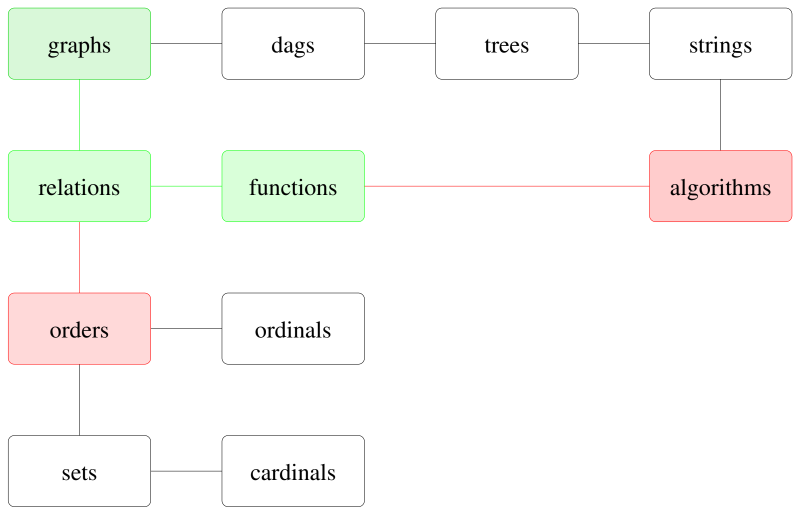
Proof.

as for Floyd ∎

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Functions as relations

Definition

A function on M is a relation R on M such that

1. for all $x \in M$, there exists y such that $x \; R \; y$ (totality)
2. for all $x, y, y' \in M$ if $x \; R \; y$ and $x \; R \; y'$ then $y = y'$, i.e. R relates uniquely.

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Example

- The squaring function on natural numbers is the relation $\{(0,0),(1,1),(2,4),(3,9),(4,16),\dots\}$.
- Taking the square root is not a function on natural numbers, since, e.g., the square root of 2 is not a natural number (existence fails)
- Taking the square root is not a function on the real numbers either, since, e.g., both -2 and 2 are square roots of 4 (uniqueness (also) fails)

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Specification of functions

A function is said to be defined by some specification this expresses that there exists a unique relation satisfying the specification and the relation is a function.

Example

The function f on natural numbers defined by

- $f(n) = n$? $\checkmark$or $f(n) = -1$? $\times$or $f(n) = f(n)$? $\times$
- $f(0) = 10$ and $f(1) = 2$? $\times$or $f(0) = 0$ and $f(n+1) = f(n)$? $\checkmark\dots$

How to represent functions?

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Problem

Domains are typically infinite. Then representation by

- list of tuples? infinitely many tuples!
- adjacency matrix? infinitely many rows,columns!
- ...?

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Solution

Represent functions by algorithms. Many ways of defining algorithms, e.g. as Turing machines, Java programs, Haskell programs, ... but all equivalent

Limitations of algorithms

- There are more functions $f : \mathbb{N} \to \mathbb{N}$ than there are algorithms (programs, TMs); so some functions cannot be represented by algorithms;

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Remark

These limitations will be addressed in the last few weeks of course

Functions as imperative programs

Definition

A deterministic 1-tape Turingmachine (TM) M is a 9-tuple

$$M = (Q, \Sigma, \Gamma, \vdash, \sqcup, \delta, s, t, r)$$

1. Q is a finite set of states
2. Σ is a set of input symbols
3. Γ is a finite set of tape symbols with $\Sigma \subseteq \Gamma$
4. $\vdash \in \Gamma \setminus \Sigma$ is the left-end marker symbol
5. $\sqcup \in \Gamma$ ($\sqcup \neq \vdash$), is the blank symbol
6. $\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function
7. $s \in Q$, the initial or start state
8. $t \in Q$, the accepting state
9. $r \in Q$, the rejecting state ($t \neq r$)

Transition function

$\delta(p, a) = (q, b, d)$ means that if TM M is in state p and reads symbol a, then

1. M replaces a with b on the tape
2. the read/write-head moves one step in direction d
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$$\forall\, p \in Q, \exists\, q \in Q \quad \delta(p, \vdash) = (q, \vdash, \mathrm{R})$$

- If TM is in the accepting/rejecting state, it will stay in that state

$$\forall b \in \Gamma \;\; \pi_1(\delta(t, b)) = t \text{ and } \pi_1(\delta(r, b)) = r$$

Here π_1 denotes projection on the first component

Example

Let $M = (\{s, p, t, r\}, \{0, 1\}, \{\vdash, \sqcup, 0, 1\}, \vdash, \sqcup, \delta, s, t, r)$ be a TM. Then δ can be specified by a transition table

	$\vdash$	0	1	$\sqcup$
s	$(s, \vdash, \mathrm{R})$	$(s, 0, \mathrm{R})$	$(s, 1, \mathrm{R})$	$(p, \sqcup, \mathrm{L})$
p	$(t, \vdash, \mathrm{R})$	$(t, 1, \mathrm{L})$	$(p, 0, \mathrm{L})$	$\cdot$

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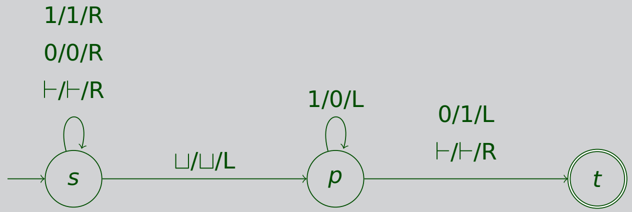
p	$(t, \vdash, \mathrm{R})$	$(t, 1, \mathrm{L})$	$(p, 0, \mathrm{L})$	$\cdot$

or by a transition graph



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- $\mathbb{B} = \{0, 1\}$ is the binary alphabet
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Remark

As in ETI, we omit parentheses in words; words of length 1 are denoted by symbols

Definition

A configuration of a TM M is a triple (p, x, n) comprising

1. $p \in Q$ its state
2. $x = y\sqcup^\infty$ its tape content, $y \in \Gamma^*$
3. $n \in \mathbb{N}$ the position of the read/write-head

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Example

For the TM M of the previous example

$$(s, \vdash 0010\sqcup^\infty, 0) \xrightarrow[M]{*} (t, \vdash 0011\sqcup^\infty, 3)$$

The step function of a TM

Definition

Let $n \in \mathbb{N}$ and $y = y_0 \cdots y_{m-1} \in \Gamma^*$ with $m > n$; the relation $\xrightarrow[M]{1}$ is defined by:

$$(p, y\sqcup^\infty, n) \xrightarrow[M]{1} \begin{cases} (q, y_0 \dots b \dots y_{m-1}\sqcup^\infty, n-1) & \delta(p, y_n) = (q, b, \mathrm{L}) \\ (q, y_0 \dots b \dots y_{m-1}\sqcup^\infty, n+1) & \delta(p, y_n) = (q, b, \mathrm{R}) \end{cases}$$

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Definition

$\xrightarrow[M]{*}$ is the reflexive–transitive closure of $\xrightarrow[M]{1}$. That is, it corresponds to a finite sequence of consecutive steps.

Definition

a TM M

- accepts $x \in \Sigma^*$, if $\exists\, y, n$:
$$(s, \vdash x\sqcup^\infty, 0) \xrightarrow[M]{*} (t, y, n)$$
- rejects $x \in \Sigma^*$, if $\exists\, y, n$:
$$(s, \vdash x\sqcup^\infty, 0) \xrightarrow[M]{*} (r, y, n)$$
- halt on input x, if x is accepted or rejected
- does not halt in input x, if x is neither accepted nor rejected
- is total, if M halts on all inputs

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Definition

A function $f : A \to B$ is defined by a TM M for every $x \in A$, M accepts input x with $f(y)$ on the tape (and does not halt or rejects on inputs $x \notin A$).

Functions as functional programs

Example

Squaring $sq : \mathbb{N} \to \mathbb{N}$

$$sq\, n = n \cdot n$$

ok, but multiplication $\cdot$? infinite table?

Functions as functional programs

Example

Squaring $sq : \mathbb{N} \to \mathbb{N}$ and multiplication $\cdot : \mathbb{N} \times \mathbb{N} \to N$

$$\begin{aligned} sq\, n &= n \cdot n \\ 0 \cdot k &= 0 \\ (1+n) \cdot k &= k + (n \cdot k) \end{aligned}$$

ok, but addition $+$? infinite table?

Functions as functional programs

Example

Squaring $sq : \mathbb{N} \to \mathbb{N}$, multiplication $\cdot : \mathbb{N} \times \mathbb{N} \to N$ and addition $+ : \mathbb{N} \times \mathbb{N} \to N$

$$\begin{aligned} sq\, n &= n \cdot n \\ 0 \cdot k &= 0 \\ (1+n) \cdot k &= k + (n \cdot k) \\ 0 + k &= k \\ (1+n) + k &= 1 + (n + k) \end{aligned}$$

ok, but successor $1+$? infinite table? (Sheet 3)

Algorithms, TMs, programs vs. functions

- finite representations of functions
- finiteness entails not all functions can be represented (later)
- a function that can be represented is called computable
- different programs may represent the same function; e.g. mergesort vs. bubblesort
- programs can be distinguished by their usage of resources; time, space, steps, energy, . . . ; functions cannot

Orders

Definition

A relation is a

- partial order if it is reflexive, anti-symmetric and transitive;
- total order if moreover every pair of elements is related either way; and
- strict order if it is irreflexive and transitive

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Strict orders are anti-symmetric

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Lemma

Strict orders are anti-symmetric

Proof.

We show anti-symmetry vacuously holds, by showing that its assumption is never fulfilled. Suppose both $x\ R\ y$ and $y\ R\ x$ for some strict order R. By transitivity, then $x\ R\ x$, contradicting irreflexivity. ■

Example

The natural order $\leq$ on $\mathbb{Z}$, defined by

$$x \leq y \quad \text{if} \quad y - x \in \mathbb{N}$$

is a partial order and a total order, but not a strict order (consider $(5, 5)$).

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Let, for tuples $x = (x_1, x_2, \ldots, x_k)$ und $y = (y_1, y_2, \ldots, y_k)$ in M^k

$$x \; R_{\text{comp}} \; y \quad \text{if} \quad x_i \; R \; y_i \quad \text{for all } i = 1, \ldots, k$$

The componentwise extension R_{comp} of R is a partial resp. strict order, if R is on M.
Typically not total, even if R is; $(2, 1) \not\leq (1, 2)$ on $\mathbb{N}^2$.

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Definition (Refining and coarsening of partitions)

Let P, Q be partitions of M. That is, $\bigcup P = M$ and $\forall p \neq p' \in P, p \cap p' = \emptyset$.

$$P \leq Q :\Leftrightarrow \text{every block of } P \text{ is a subset of a block of } Q$$

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Example

The partition $\{\{a\}, \{b\}, \{c\}\}$ is (strictly) finer than each of

$$\{\{a\}, \{b,c\}\} \quad \{\{b\}, \{a,c\}\} \quad \{\{c\}, \{a,b\}\} \quad \{\{a,b,c\}\}$$

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(1) *If $\leq$ partial order, then its predecessor relation*

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(3) *The functions $\leq \mapsto <$ in (1) and $< \mapsto \leq$ in (2) are inverse to each other*

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Partial orders defined by strict orders, and vice versa

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(1) By definition, $x < y$ holds iff $x \leq y$ and $x \neq y$. Therefore, $<$ is irreflexive.

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(3) Starting from a partial order $\leq$, the relation defined by $(x \leq y \,\wedge\, x \neq y) \,\vee\, x = y$ is $\leq$ again, as the construction amount to first removing all loops, and then adding them, noting $\leq$ has all loops. The other direction is similar, using that a strict order $<$ has no loops.

Definition

Let $\leq$ be a partial order on M. Then $x \in M$ is

- least in M, if for all $y \in M$, $x \leq y$
- greatest in M, if for all $y \in M$, $y \leq x$
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Example

$\leq$ generated by predecessor relation

$$< = \{(1,2),(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$$

- minimal elements:
- maximal elements:
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$$< = \{(1,2),(1,4),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\}$$

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