Summary last week

- functions as algorithms; finite specifications
- functions defined by imperative programs
- Turing machines; input and output on tape, transitions, halting
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- functions as algorithms; finite specifications
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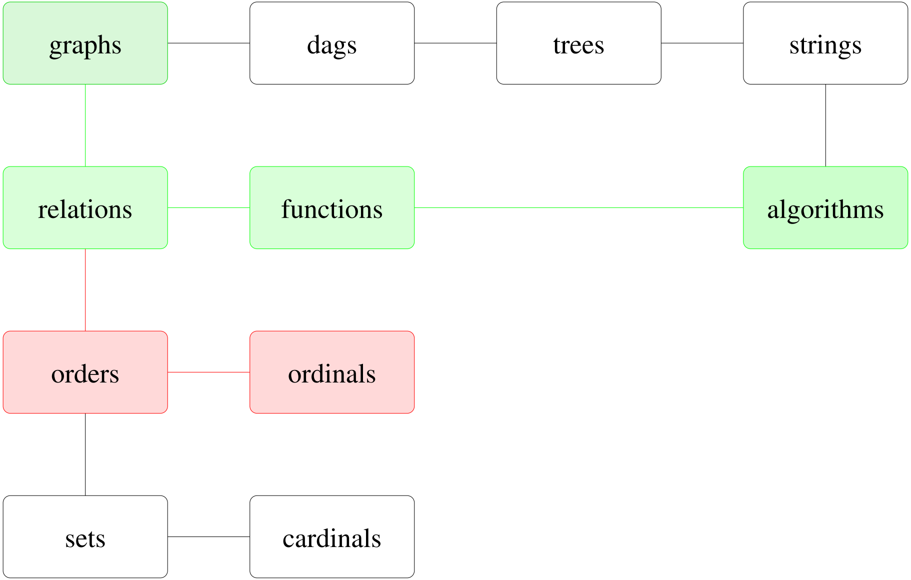
- functional specifications; input as argument(s), output as value, replacing
- orders as certain transitive relations; partial, total, strict
- correspondence between partial and strict orders
- strict part (predecessor): $\leq \mapsto <$; reflexive closure: $< \mapsto \leq$
- minimal/maximal elements: no element smaller/greater
- least/greatest elements: smaller/greater than all

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Orders

Definition

A relation is a

- partial order if it is reflexive, anti-symmetric and transitive;
- total order if moreover every pair of elements is related either way; and
- strict order if it is irreflexive and transitive (so it is anti-symmetric)

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Example

The natural order $\leq$ on $\mathbb{Z}$, defined by $x \leq y$ if $y - x \in \mathbb{N}$ is partial, total order (not strict). $<$ is strict (not total, partial).

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$m \in \mathbb{N}$ divides $n \in \mathbb{N}$, if there is some $p \in \mathbb{N}$ such that $n = m \cdot p$. Divisibility is a partial order (not total, strict). Strict divisibility is strict (not total, partial).

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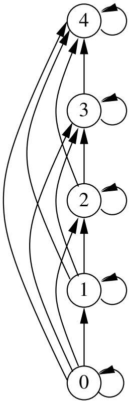
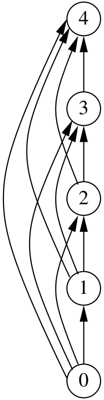
Partial order ⇒ strict order ⇒ Hasse diagram



(initial part of) graph of partial order $\leq$ on $\mathbb{N}$

why have reflexive, transitive edges if we can reconstruct them?

Partial order $\Rightarrow$ strict order $\Rightarrow$ Hasse diagram



graph of strict order $<$ on $\mathbb{N}$

$\leq$ reconstructed from strict order as reflexive closure $<^=$ of $<$

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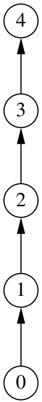

graph of successor relation $R = \{(n, n+1) \mid n \in \mathbb{N}\}$; Hasse diagram of $\leq$

$\leq$ reconstructed from Hasse diagram as reflexive–transitive closure R^* of R

Lemma

$\leq$ *total order*

- *x least* $\Leftrightarrow$ *x minimal*
- *x greatest* $\Leftrightarrow$ *x maximal*

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minimal: x least and $y \leq x \Rightarrow y \leq x \leq y \Rightarrow y = x$
(2) By (1) using that greatest, maximal wrt $\leq$ iff least, minimal wrt its converse $\geq$ ∎

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(3) *M finite $\Rightarrow$ for every $x \in M$ there exist a minimal w such that $w \leq x$ and a maximal z such that $x \leq z$*

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(4) and (5) follow from (3) ■

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Definition (Alphabet)

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- $\mathbb{B} = \{0, 1\}$ is the binary alphabet
- $\{a, b, \ldots, z\}$ is the alphabet of letters
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Let $\Sigma = \{a, b\}$ and $a < b$. Then

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$\leq_{lex}$ is a partial, total order on Σ^*

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Then we have for $m := \min(k, l)$, $m \leq \ell(u)$ and $m \leq \ell(w)$ and

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from which $u <_{\text{lex}} w$ follows

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Since $\leq$ is total on Σ, we have either $v <_{\text{lex}} w$ or $w <_{\text{lex}} v$

Well-founded relations

Definition (well-founded relation)

- Let R be a relation on a set M
- A sequence $(x_0, x_1, x_2, \ldots)$ of elements of M is an infinite descending R-chain, if
$$\ldots\ R\ x_2\ R\ x_1\ R\ x_0$$
- R is well-founded, if M has no infinite descending R-chains.
- When we say that partial order $\leq$ is well-founded we mean that its strict part $<$ is

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Example

- The natural order $\leq$ on $\mathbb{N}$ is well-founded
- The natural order $\leq$ on $\mathbb{Z}$ is not well-founded
- The lexicographic order is not well-founded, if alphabet has at least two symbols

Proving that all elements of set have some property

Universal properties

Given: M a set and P a property of elements of the set

Goal: establish that all elements of M have property P

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- $M =$ months of year; $P(m) =$ month m has at least 25 days

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- $M =$ pairs of positive natural numbers, $P(n, m) =$ Euclid's algorithm yields $\gcd(m, n)$

Proof by cases

Program

```
data Month = Jan | Feb | Mar | Apr | May | Jun
           | Jul | Aug | Sep | Oct | Nov | Dec
days :: Month -> Int
days Jan = 31
...
days Dec = 31
```

Lemma

for every `Month m`, `days m` ≥ 25

Proof by cases

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Lemma

for every `Month m`, `days m` ≥ 25

Proof by cases.

`days Jan = 31` ≥ 25 ✓, . . . , `days Dec = 31` ≥ 25 ✓
we conclude since we checked all cases ■

Proof by universal generalisation

Lemma

for every natural number n that is even, n^2 is even.

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2. suppose n is even: $n = 2m$ for some natural number m

Proof by universal generalisation

Lemma

for every natural number n that is even, n^2 is even.

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we conclude since n was taken to be arbitrary ∎

Proof by mathematical induction

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for every natural number $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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Principle of well-founded induction

Assumption: R a well-founded relation on set N

Induction: for arbitrary $n \in N$, show that if $P(m)$ for all $m \mathrel{R} n$, then $P(n)$

Conclude: for all $n \in N$, $P(n)$

the $P(m)$ for $m \mathrel{R} n$ are the induction hypotheses

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Proof.

- Take the well-founded relation $\{(n, n+1) \mid n \in \mathbb{N}\}$.
- if $n = 0$, then no induction hypotheses; directly show $P(0)$

$$\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2}$$

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Proof.

- Take the well-founded relation $\{(n, n+1) \mid n \in \mathbb{N}\}$.
- if $n > 0$, then one induction hypothesis $P(n-1)$: $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$

$$\sum_{i=1}^{n} i = (\sum_{i=1}^{n-1} i) + n =_{IH} \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2}$$

Proof by mathematical induction

mathematical induction

1. Suppose we want to show $P(n)$ for all natural numbers n
2. To that end, we may proceed as follows:
   - Induction basis: We show that P holds for the base value 0;
   - Induction step: We show that for all $n > 0$, $P(n-1)$ entails $P(n)$.
3. Then $P(n)$ holds for all n

Mathematical induction = well-founded induction wrt. $R = \{(n, n+1) \mid n \in \mathbb{N}\}$.

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$$(P(0) \wedge \forall n > 0.(P(n-1) \rightarrow P(n))) \rightarrow (\forall n.P(n))$$

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well-founded induction formally

$$\forall n.((\forall m \text{ such that } m\ R\ n.P(m)) \to P(n)) \to (\forall n.P(n))$$

Foundations of well-founded induction

Theorem

Let $\leq$ be a partial order on the set M. Then $\leq$ is well-founded iff every non-empty subset of M has a minimal element.

Proof.

Let $\leq$ be a well-founded order on M and N a non-empty subset of N. Then there exists some element x_0 in N. If x_0 is minimal in N, then we are done.

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Otherwise, there exists some element $x_1 \in N$ with $x_1 < x_0$. If x_1 is minimal, then we are done again. Otherwise, there is some $x_2 \in N$ with $x_2 < x_1$, etc.. Since

$$x_0 > x_1 > x_2 > \ldots$$

we reach a minimal element x_n after finitely many steps.

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To prove the other direction, we suppose that $\leq$ were not well-founded. Then there would be an infinitely descending chain

$$x_0 > x_1 > x_2 > \dots ,$$

and the non-empty subset $N = \{x_0, x_1, x_2, \dots\}$ has no minimal element.

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Proof by well-founded induction

Lemma

for all pairs of positive natural numbers, Euclid's algorithm yields $\mathrm{gcd}(m, n)$

Euclid's greatest common divisor algorithm

```
euclid m n = if m == n then m else if m > n
  then euclid (m-n) n else euclid m (n - m)
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- Take the well-founded relation $\{((m, n), (m', n')) \mid m + n < m' + n'\}$.

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- if $m = n$, then no induction hypotheses needed; `euclid m m` $= m = \gcd(m, m)$
- if $m > n$, then induction hypotheses: `euclid m' n'` $= \gcd(m', n')$ if $m' + n' < m + n$

  `euclid m n` $=$ `euclid (m-n) n` $=_{IH} \gcd(m - n, m) = \gcd(m, n)$ ∎

Example

Let M be the set of all palindromes over the alphabet $\{a, b\}$. We show

?*If $x \in M$ and $\ell(x)$ even, then x has an even number of as.****?***

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Let M be the set of all palindromes over the alphabet $\{a, b\}$. We show

?If $x \in M$ and $\ell(x)$ even, then x has an even number of as.?

Proof.

By well-founded induction. Take $R = \{(w, w') \mid \ell(w) < \ell(w')\}$; order by length

- if x the empty string, then property holds; 0 is even
- if x non-empty induction hypotheses: property holds for words shorter than x
  - if first letter of x is a, then $x = ax'a$ for some palindrome $x' \in M$. then conclude since $2 +$ even is even
  - if first letter of x is b, then $x = bx'b$ for some palindrome $x' \in M$. then conclude since $0 +$ even is even

■