# Summary last week

- Hasse diagram of a partial order $\leq$ or strict order $<$
- least irreflexive, atransitive subrelation $R$ of $\leq$ such that $\leq = R^*$ or $< = R^+$ (atransitive: $x \; R \; y$ and $y \; R \; z$ then not $x \; R \; z$)
- for total orders, minimal = least and maximal = greatest
- finite partial orders have minimal and maximal elements
- the lexicographic order $<_{\text{lex}}$ on words; partial/total if $\leq$ is.

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- for total orders, minimal = least and maximal = greatest
- finite partial orders have minimal and maximal elements
- the lexicographic order $<_{\text{lex}}$ on words; partial/total if $\leq$ is.
- well-founded relations as not having infinite descending chains
- Three methods to prove that all elements of set have some property:

**1)** by cases; for finite sets, enumerating all elts

**2)** by universal generalisation; for infinite sets, proving for some arbitrary elt

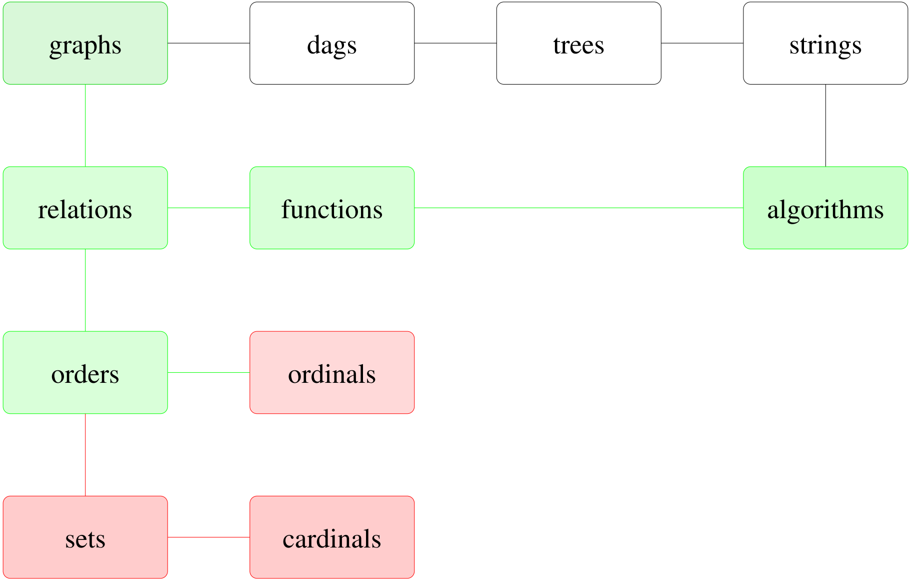
**3)** by well-founded induction; for infinite sets, using property (IH) for smaller elts

- well-founded induction principle for well-founded relation $R$, property $P$: $\forall n.((\forall m \text{ such that } m\ R\ n.P(m)) \rightarrow P(n)) \rightarrow (\forall n.P(n))$

# Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

# Discrete structures



# Well-founded relations

**Definition (well-founded relation)**

- Let $R$ be a relation on a set $M$
- A sequence $(x_0, x_1, x_2, \ldots)$ of elements of $M$ is an infinite descending $R$-chain, if

$$\ldots R\ x_2\ R\ x_1\ R\ x_0$$

- $R$ is well-founded, if $M$ has no infinite descending $R$-chains.
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## Principle of well-founded induction

Assumption: $R$ a well-founded relation on set $N$. $P$ a property of $n \in N$.
Induction: for arbitrary $n \in N$, show that if $P(m)$ for all $m$ such that $m\ R\ n$, then $P(n)$
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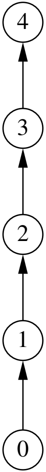
# Mathematical induction

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P(3)?

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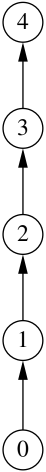


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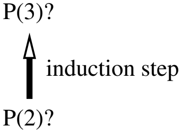

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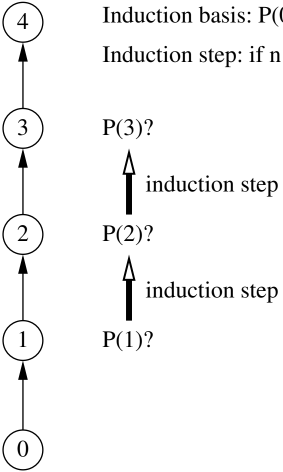

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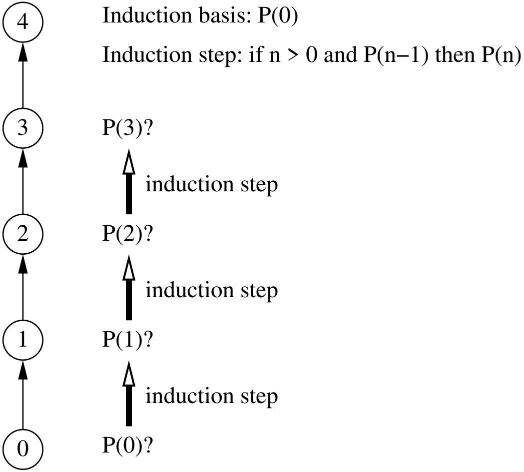

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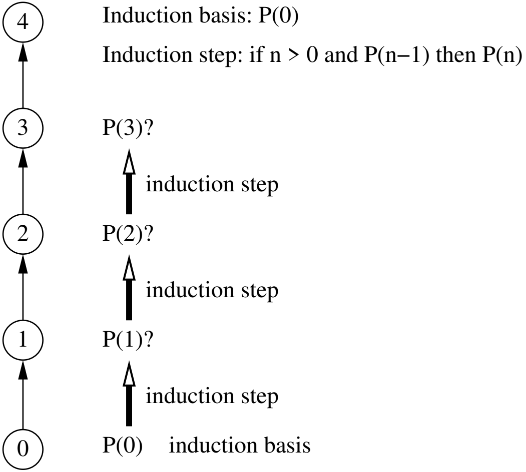

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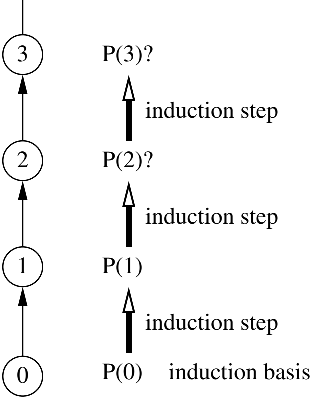

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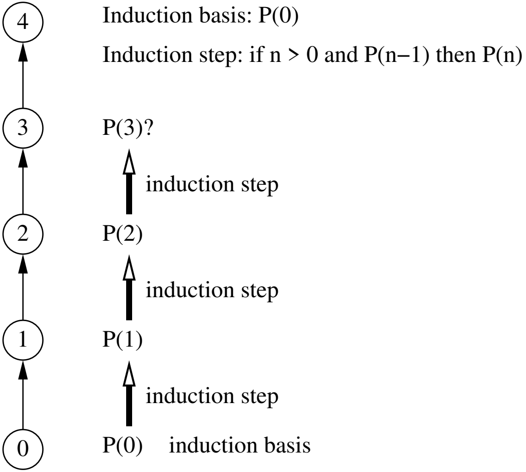

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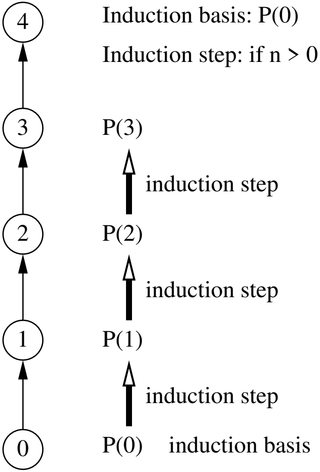

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- on inductively defined structures: sub-structure

# Experimenting to find what may be helpful

**Example**

Let $M$ be the set of all palindromes over the alphabet $\{a, b\}$. To show $\forall x \in M.P(x)$ where $P(x) =$ if $\ell(x)$ even, then $x$ has an even number of $a$s.

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  - if first letter of $x$ is $a$, then $x = ax'a$ for some $x' \in M$ of even length. By the IH $P(x')$ holds, i.e. $x'$ has an even number of $a$s. 2 + even is even.
  - if first letter of $x$ is $b$, then $x = bx'b$ for some $x' \in M$ of even length. $0+$ even is even.

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## Ackermann function

Function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?

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ack 0 n = n+1
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well-founded relation such that all of these are smaller?

# Lexicographic product

**Definition**

Let $\leq_1, \leq_2$ be partial orders. Their lexicographic product is defined by

$$(x_1, x_2) \quad \leq_1 \times_{\text{lex}} \leq_2 \quad (y_1, y_2)$$

if $x_1 <_1 y_1$ or ($x_1 = y_1$ and $x_2 \leq_2 y_2$).

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**Remarks**

First compare the first elements; if that does not decide compare the second elements.
Case $\leq_1 = \leq_2$ corresponds to lexicographic order restricted to strings of length 2.

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**Example**

Computing `ack m n` always yields a (unique) value, because recursive calls have arguments that are strictly smaller w.r.t. $\leq \times_{\text{lex}} \leq$. That is, we may speak of the Ackermann function.

# Inductively defined structures

**Inductive definitions**

A set $S$ of structures is inductively defined by clauses of shape

$$\text{if } n_1, \ldots, n_k \in N, \text{ then } s(n_1, \ldots, n_k) \in N$$

with $s$ structures depending on $n_1, \ldots, n_k$. if it is the least set satisfying the clauses.

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## Proof.

By $S$ being least, every element in $S$ has a unique and finite derivation tree with nodes in $S$, and only constructed from clauses (with leaves constructed by the base-clauses ($k = 0$)). $R$ relates children to parents in the tree. ∎

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The natural numbers $\mathbb{N}$ can be inductively defined by:

- $0 \in N$
- if $n \in N$, then $n + 1 \in N$.

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The palindromes $P$ over $\{0, 1\}$ can be inductively defined by:

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Sub-structure relation: ‘middle’-sub-palindromes

## Definition (inductive with explicit base cases )

A set $M$ can be defined inductively by:

- Induction basis: We introduce one or more elements of $M$.
- Induction step: We specify how, on the basis of already constructed elements of $M$, new elements of $M$ can be constructed

The set $M$ then comprises exactly those elements that can be obtained by the repeated application of the induction step on elements constructed by the induction basis (finitely many only; corresponding to least)

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## Example

The formulas of propositional logic may be inductively defined by:

1. An atomic formula $p$ is a formula
2. A truth symbol (True, False) is a formula
3. If $A$ and $B$ are formulas, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$

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**Theorem**

*Let $\leq$ be a partial order on the set $M$. Then $\leq$ is well-founded iff every non-empty subset of $M$ has a minimal element.*

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Then $n = m \cdot k$ with $m, k < n$. By minimality $m$ and $k$ are products of primes, but then so is $n$. Contradiction.

# Questions and methodology for structures

- When are two structures the same?
- When is one structure a sub-structure of another?
- How can we represent structures?
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- What operations can we do on the structures?

For relations, functions, partial orders, well-founded relations.

# Preservation

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Positiveness is preserved by addition and multiplication. Negativeness is preserved by addition but not by multiplication.

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**Lemma**

*The componentwise extension preserves well-foundedness, i.e. if $\leq$ is a well-founded partial order, then so is $\leq_{comp}$.*

**Proof.**

For a proof by contradiction, suppose $x_1 >_{\text{comp}} x_2 >_{\text{comp}} x_3 >_{\text{comp}} \dots$ were an infinite descending $\leq_{\text{comp}}$-chain, where $x_i = (x_{i1}, \dots, x_{ik})$, for some minimal $k$. Then for their first elements $x_{11} \geq x_{21} \geq x_{31} \geq \dots$. Either this contains an infinite descending $\leq$-chain, or there exists an $N$ such that for all $n \geq N$, $x_{N1} = x_{n1}$ and then $(x_{N2}, \dots, x_{Nk}) >_{\text{comp}} (x_{N+1\,2}, \dots, x_{N+1\,k}) >_{\text{comp}} (x_{N+2\,2}, \dots, x_{N+2\,k}) >_{\text{comp}} \dots$ would be an infinite descending chain for a smaller $k$. Contradiction. ∎

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- product $(a, a') \mathrel{R \times S} (b, b')$ if $a \mathrel{R} b$ and $a' \mathrel{S} b'$;
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**Definition**

A relation $R$ is

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- a well-order if moreover for all $a$, $b$, $a \; R \; b$ or $a = b$ or $b \; R \; a$ holds

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This extends to partial orders $\leq$ via their strict part $<$.

**Theorem**

*A relation is a well-founded order iff it is a well-founded strict order.*

**Proof.**

It suffices to show that a well-founded transitive relation $R$ is irreflexive. This holds, since if $a \; R \; a$ were to hold, then $\ldots R \; a \; R \; a \; R \; a$ would be an infinite descending chain, contradicting well-foundedness. ∎

# Examples of well-(founded )orders

**Example**

Less-than is a well-order on the natural numbers, but greater-than is not (not well-founded), and neither is $\{(n, n+1) \mid n \in \mathbb{N}\}$ (not transitive).

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**Example**

The prefix order is a well-founded order on the strings over $\Sigma$, but not a well-order in case $\Sigma$ as more than 1 symbol (neither of $ab$, $ba$ is a prefix of the other).

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**Motivation/intuition**

Capture ordinals as in counting; e.g. the 1st, the 2nd, the 100th.

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Well-orders $<$ on $A$ and $\sqsubset$ on $B$ are isomorphic if there is a bijection $f$ from $A$ to $B$ with

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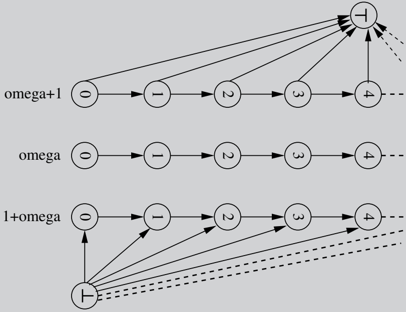
**Example**

Each finite well-order isomorphic to $<$ on $\{m \mid m < n\}$ for some $n \in N$.

# Infinite ordinals

**Example**

Extending the ordinal $\omega$ of the natural numbers either with an element $\bot$ smaller than all natural numbers to $1 + \omega$, or with an element $\top$ greater than all natural numbers to $\omega + 1$, can be depicted (omitting many transitive arrows) as:




we see that $\omega$ and $1 + \omega$ are isomorphic, but non-isomorphic to $\omega + 1$.

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If there exists a bijection $f: M \to N$, then the sets $M$ and $N$ are equinumerous or equipollent. Cardinals represent equinumerous sets.

**Example**

Each finite set equinumerous to set $\{m \mid m < n\}$ for some $n \in \mathbb{N}$.

**Example**

Adjoining $*$ to the natural numbers is equinumerous to the natural numbers; $\omega$, $1 + \omega$, and $\omega + 1$ are equinumerous as sets of nodes (forgetting about the edges/order).