Summary last week

- Hasse diagram of a partial order $\leq$ or strict order $<$
- least irreflexive, atransitive subrelation R of $\leq$ such that $\leq = R^*$ or $< = R^+$ (atransitive: $x\ R\ y$ and $y\ R\ z$ then not $x\ R\ z$)
- for total orders, minimal = least and maximal = greatest
- finite partial orders have minimal and maximal elements
- the lexicographic order $<_{\text{lex}}$ on words; partial/total if $\leq$ is.

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- for total orders, minimal = least and maximal = greatest
- finite partial orders have minimal and maximal elements
- the lexicographic order $<_{\text{lex}}$ on words; partial/total if $\leq$ is.
- well-founded relations as not having infinite descending chains
- Three methods to prove that all elements of set have some property:

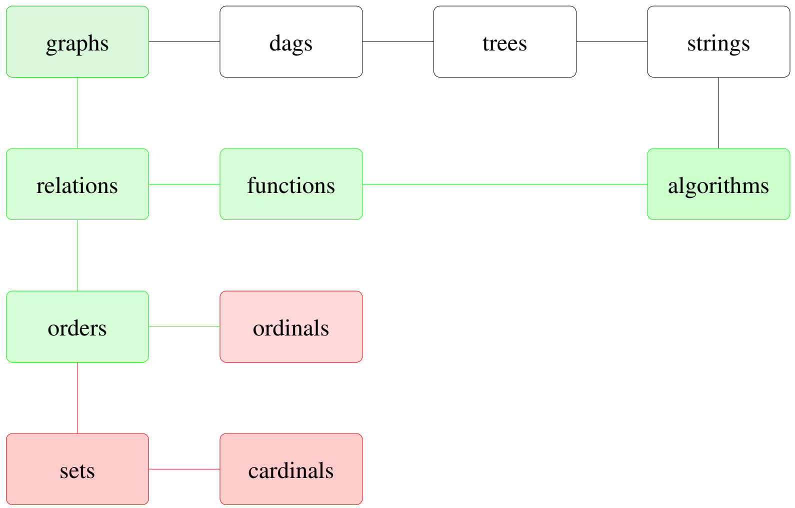
1) by cases; for finite sets, enumerating all elts
2) by universal generalisation; for infinite sets, proving for some arbitrary elt
3) by well-founded induction; for infinite sets, using property (IH) for smaller elts

- well-founded induction principle for well-founded relation R, property P: $\forall n.((\forall m \text{ such that } m\ R\ n.P(m)) \rightarrow P(n)) \rightarrow (\forall n.P(n))$

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Well-founded relations

Definition (well-founded relation)

- Let R be a relation on a set M
- A sequence $(x_0, x_1, x_2, \ldots)$ of elements of M is an infinite descending R-chain, if

$$\ldots R\ x_2\ R\ x_1\ R\ x_0$$

- R is well-founded, if M has no infinite descending R-chains.
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Principle of well-founded induction

Assumption: R a well-founded relation on set N. P a property of $n \in N$.
Induction: for arbitrary $n \in N$, show that if $P(m)$ for all m such that $m\ R\ n$, then $P(n)$
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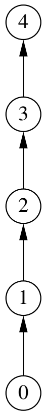
Mathematical induction



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P(3)?

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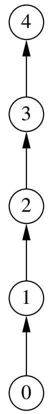

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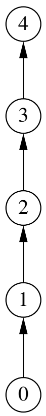

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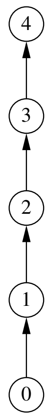

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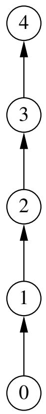

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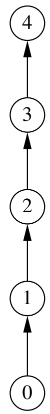

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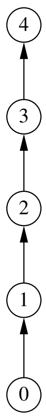

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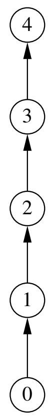

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Try to see whether, when proving $P(n)$ for an arbitrary element $n \in N$, it may be helpful to know that $P(m)$ holds already for m smaller than n, for an appropriate notion of smaller. For instance:

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- on inductively defined structures: sub-structure

Experimenting to find what may be helpful

Example

Let M be the set of all palindromes over the alphabet $\{a, b\}$. To show $\forall x \in M.P(x)$ where $P(x) =$ if $\ell(x)$ even, then x has an even number of as.

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  - if first letter of x is b, then $x = bx'b$ for some $x' \in M$ of even length. 0+ even is even.

Experimenting with the Ackermann function

Ackermann function

Function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?

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ack 0 n = n+1
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well-founded relation such that all of these are smaller?

Lexicographic product

Definition

Let $\leq_1, \leq_2$ be partial orders. Their lexicographic product is defined by

$$(x_1, x_2) \quad \leq_1 \times_{\text{lex}} \leq_2 \quad (y_1, y_2)$$

if $x_1 <_1 y_1$ or ($x_1 = y_1$ and $x_2 \leq_2 y_2$).

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Remarks

First compare the first elements; if that does not decide compare the second elements.
Case $\leq_1 = \leq_2$ corresponds to lexicographic order restricted to strings of length 2.

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Example

Computing `ack m n` always yields a (unique) value, because recursive calls have arguments that are strictly smaller w.r.t. $\leq \times_{\text{lex}} \leq$. That is, we may speak of the Ackermann function.

Inductively defined structures

Inductive definitions

A set S of structures is inductively defined by clauses of shape

$$\text{if } n_1, \ldots, n_k \in N \text{, then } s(n_1, \ldots, n_k) \in N$$

with s structures depending on $n_1, \ldots, n_k$. if it is the least set satisfying the clauses.

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If $s(n_1, \ldots, n_k) \in S$, then $n_1, \ldots, n_k \in S$, assuming the former uniquely depends on the latter, and then the sub-structure relation is well-founded.

Proof.

By S being least, every element in S has a unique and finite derivation tree with nodes in S, and only constructed from clauses (with leaves constructed by the base-clauses ($k = 0$)). R relates children to parents in the tree. ∎

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The natural numbers $\mathbb{N}$ can be inductively defined by:

- $0 \in N$
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Example

The palindromes P over $\{0, 1\}$ can be inductively defined by:

- $\epsilon, 0, 1 \in N$
- if $n \in N$, then $0n0 \in N$, $1N1 \in N$.

Sub-structure relation: 'middle'-sub-palindromes

Definition (inductive with explicit base cases)

A set M can be defined inductively by:

- Induction basis: We introduce one or more elements of M.
- Induction step: We specify how, on the basis of already constructed elements of M, new elements of M can be constructed

The set M then comprises exactly those elements that can be obtained by the repeated application of the induction step on elements constructed by the induction basis (finitely many only; corresponding to least)

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Example

The formulas of propositional logic may be inductively defined by:

1. An atomic formula p is a formula
2. A truth symbol (True, False) is a formula
3. If A and B are formulas, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$

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Example

Assumption: $P(n) = n$ can be written as a product of primes if $n > 2$. $<$ on $\mathbb{N}$.

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Assumption: $P(n) = n$ can be written as a product of primes if $n > 2$. $<$ on $\mathbb{N}$.
Minimal counterexample: let n be minimal and not a product of primes.

Proof by minimal counterexample

Theorem

Let $\leq$ be a partial order on the set M. Then $\leq$ is well-founded iff every non-empty subset of M has a minimal element.

Proof by minimal counterexample

Assumption: P a property on set N. R a well-founded relation on N
Minimal counterexample: show if n minimal such that not $P(n)$, then contradiction
Conclude: for all $n \in N$, $P(n)$

Example

Assumption: $P(n) = n$ can be written as a product of primes if $n > 2$. $<$ on $\mathbb{N}$.
Minimal counterexample: let n be minimal and not a product of primes.
Then $n = m \cdot k$ with $m, k < n$. By minimality m and k are products of primes, but then so is n. Contradiction.

Questions and methodology for structures

- When are two structures the same?
- When is one structure a sub-structure of another?
- How can we represent structures?
- What operations can we do on the structures?

Questions and methodology for structures

- What operations can we do on the structures?

For relations, functions, partial orders, well-founded relations.

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Definition

A property P is preserved by some operation, If P holds for the arguments, then it holds for the result.

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Example

Positiveness is preserved by addition and multiplication. Negativeness is preserved by addition but not by multiplication.

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Lemma

The componentwise extension preserves well-foundedness, i.e. if $\leq$ is a well-founded partial order, then so is $\leq_{comp}$.

Proof.

For a proof by contradiction, suppose $x_1 >_{\text{comp}} x_2 >_{\text{comp}} x_3 >_{\text{comp}} \dots$ were an infinite descending $\leq_{\text{comp}}$-chain, where $x_i = (x_{i1}, \dots, x_{ik})$, for some minimal k. Then for their first elements $x_{11} \geq x_{21} \geq x_{31} \geq \dots$. Either this contains an infinite descending $\leq$-chain, or there exists an N such that for all $n \geq N$, $x_{N1} = x_{n1}$ and then $(x_{N2}, \dots, x_{Nk}) >_{\text{comp}} (x_{N+1\,2}, \dots, x_{N+1\,k}) >_{\text{comp}} (x_{N+2\,2}, \dots, x_{N+2\,k}) >_{\text{comp}} \dots$ would be an infinite descending chain for a smaller k. Contradiction. ∎

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- product $(a, a')\ R \times S\ (b, b')$ if $a\ R\ b$ and $a'\ S\ b'$;
  relation on $A \times A$

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Well-(founded)orders

Definition

A relation R is

- a well-founded order if it is well-founded and transitive
- a well-order if moreover for all a, b, $a\ R\ b$ or $a = b$ or $b\ R\ a$ holds

This extends to partial orders $\leq$ via their strict part $<$.

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Theorem

A relation is a well-founded order iff it is a well-founded strict order.

Proof.

It suffices to show that a well-founded transitive relation R is irreflexive. This holds, since if $a\ R\ a$ were to hold, then $\ldots R\ a\ R\ a\ R\ a$ would be an infinite descending chain, contradicting well-foundedness. ∎

Examples of well-(founded)orders

Example

Less-than is a well-order on the natural numbers, but greater-than is not (not well-founded), and neither is $\{(n, n+1) \mid n \in \mathbb{N}\}$ (not transitive).

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Divisibility is a well-founded order on the natural numbers: it's a partial order with its strict part well-founded. It is not a well-order.

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Example

The prefix order is a well-founded order on the strings over Σ, but not a well-order in case Σ as more than 1 symbol (neither of ab, ba is a prefix of the other).

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Motivation/intuition

Capture ordinals as in counting; e.g. the 1st, the 2nd, the 100th.

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Definition

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1. if $a < a'$ then $f(a) \sqsubset f(a')$;
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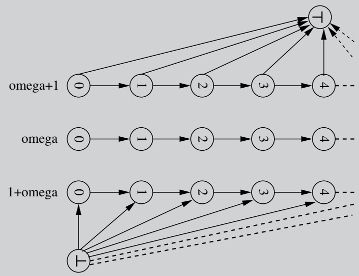
Example

Each finite well-order isomorphic to $<$ on $\{m \mid m < n\}$ for some $n \in N$.

Infinite ordinals

Example

Extending the ordinal ω of the natural numbers either with an element $\bot$ smaller than all natural numbers to $1 + \omega$, or with an element $\top$ greater than all natural numbers to $\omega + 1$, can be depicted (omitting many transitive arrows) as:



we see that ω and $1 + \omega$ are isomorphic, but non-isomorphic to $\omega + 1$.

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Definition

If there exists a bijection $f : M \to N$, then the sets M and N are equinumerous or equipollent. Cardinals represent equinumerous sets.

Example

Each finite set equinumerous to set $\{m \mid m < n\}$ for some $n \in \mathbb{N}$.

Example

Adjoining $*$ to the natural numbers is equinumerous to the natural numbers; ω, $1 + \omega$, and $\omega + 1$ are equinumerous as sets of nodes (forgetting about the edges/order).