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- inductively defined structures as least set satisfying clauses
- structural induction as induction principle w.r.t. sub-structure relation
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- relation operations: identity, converse, intersection, union, composition, product
- preservation of property P by n-ary operation f: $P(f(x1,\ldots,x_n))$, if $P(x_1),\ldots,P(x_n)$
- being a function preserved: identity, composition, product
- being a partial order preserved: identity, converse, intersection, (lex) product
- being well-founded preserved: intersection, (lex) product, comp. extension

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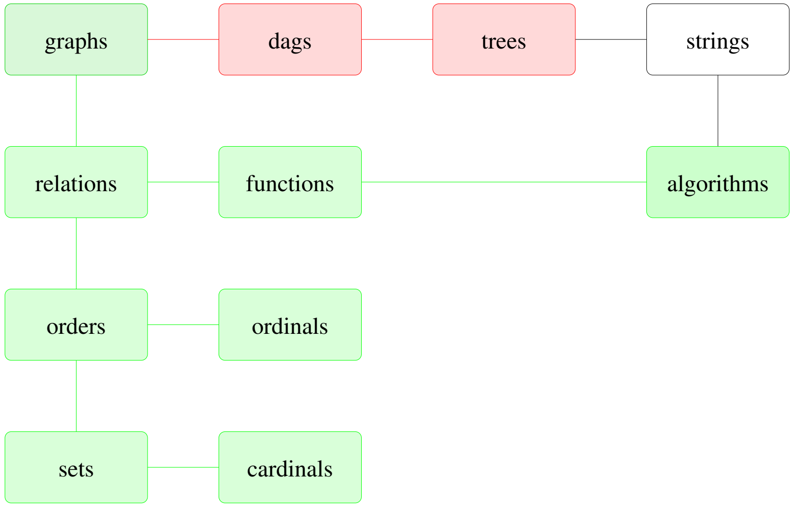
- being well-founded preserved: intersection, (lex) product, comp. extension
- well-founded /well-orders as well-founded partial/total orders
- counting by cardinals, sets w.r.t. bijection; equinumerous
- counting by ordinals, well-orders w.r.t. isomorphism; order-preserving bijection

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Dags and trees motivation

Example (Dags)

- resource dependencies (build, citation)
- statement dependencies (out-of-order execution)
- sub-expression sharing (call-by-need)
- binary decision diagrams
- ...

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Example (Trees)

- data structures (searching, sorting, XML)
- parse tree (of text)/abstract syntax tree (of program)
- spanning tree (of graph)
- computation tree (of non-deterministic machines)
- ...

Dags and trees

Definition (Cycle)

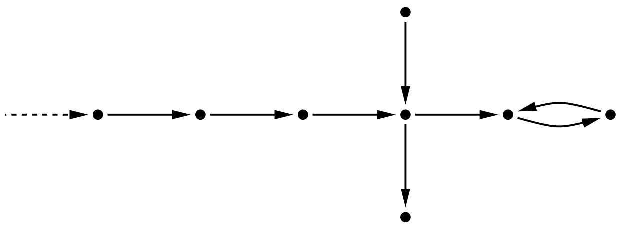
Let (V, E, src, tgt) be a directed multigraph

- a path is closed if its source is its target
- a non-empty closed path without repeated edges is a cycle
- directed multigraphs without cycles are cycle-free

Definition (Dags, forests and (rooted) trees)

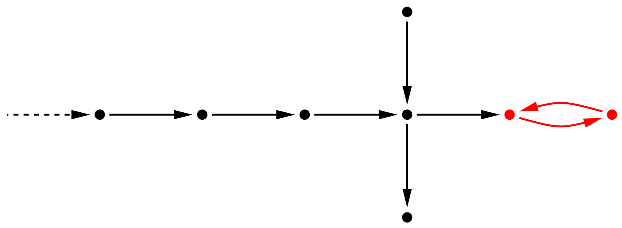
- a dag is a directed acyclic graph
- a forest is a dag with nodes of in-degree ≤ 1
- in a forest, nodes with out-degree 0 are called leaves
- a tree is a forest where all v_1, v_2 have a common ancestor v having paths to both
- a rooted tree is a tree with a node, the root, having a path to all nodes

Dags and trees example



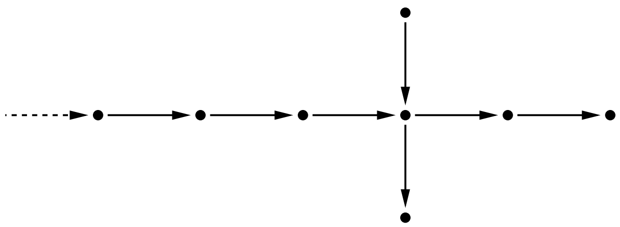
graph

Dags and trees example



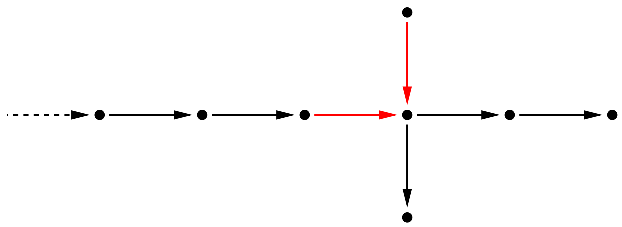
graph but not a dag (cycle)

Dags and trees example



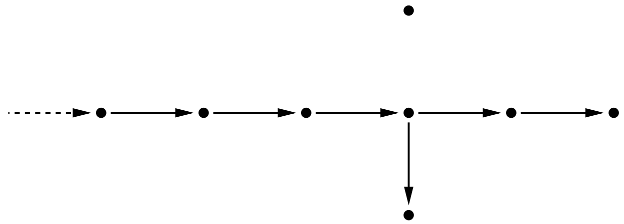
dag

Dags and trees example



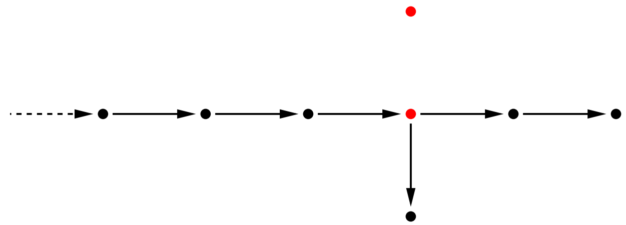
dag but not a forest (indegree 2)

Dags and trees example



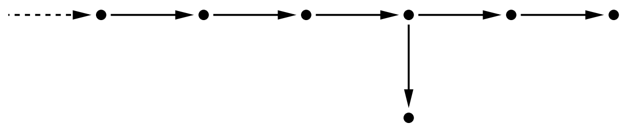
forest

Dags and trees example



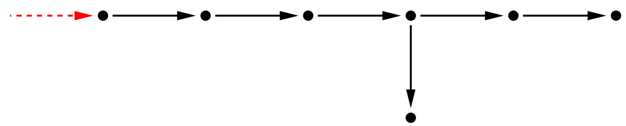
forest but not a tree (no common ancestor)

Dags and trees example



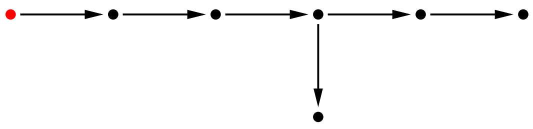
tree

Dags and trees example



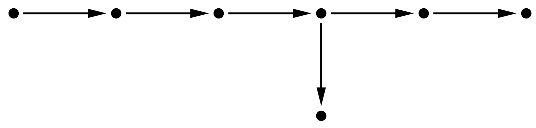
tree but not a rooted tree (no root)

Dags and trees example



rooted tree (root)

Dags and trees example



rooted tree

Simplicity in cycles

Lemma

simple paths do not have repeated edges.

Proof.

Let p be a simple path. if some edge e were to occur twice in it, the source node v of both occurrences of e would occur twice as well. ■

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every simple closed path in a multigraph is a cycle

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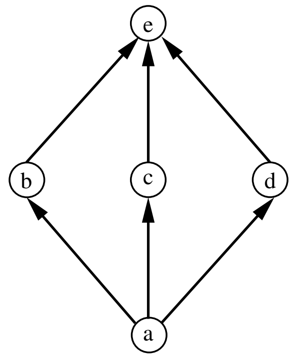
Remark

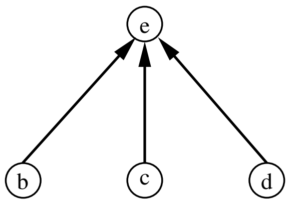
Since paths may be shortened to simple paths, cycles represent closed paths.

Topological sorting example



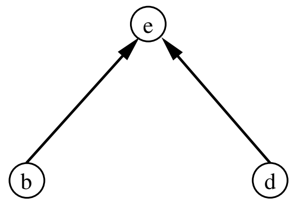
topological sorting: ()

Topological sorting example



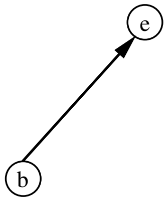
topological sorting: (a)

Topological sorting example



topological sorting: (a, c)

Topological sorting example



topological sorting: (a, c, d)

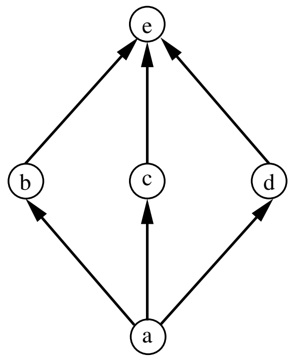
Topological sorting example

topological sorting: (a, c, d, b)

Topological sorting example

topological sorting: (a, c, d, b, e)

Topological sorting example



topological sorting: (a, c, d, b, e)
others: (a, c, b, d, e), (a, b, c, d, e), (a, b, d, c, e), (a, d, b, c, e), (a, d, c, b, e)

Topological sorting

Definition

a list $(a_0, \ldots, a_{n-1})$ is topologically $\leq$-sorted for partial order $\leq$, if $a_i < a_j$ implies $i < j$

Remark

if $\leq$ is a total order, then topologically $\leq$-sorted iff globally $\leq$-sorted

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Proof.

if $l = (a_0, \ldots, a_{n-1})$ is topologically sorted and no element of A is smaller than any element of l, then so are $l' = (a_0, \ldots, a_{n-1}, a_n)$ and $A - \{a_n\}$ for a_n minimal in A: l' is topologically sorted since l is, and $a_n \not< a_j$ since no element of A is smaller than any element of l, and if $a_i < a_n$ then $i < n$ because $0 \leq i < n$ is an index in l. ∎

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Proof.

reflexivity and transitivity hold by empty paths and composing paths. To see anti-symmetry consider paths from v to v' and from v' to v. Both must be empty as otherwise their composition would yield a cycle in G, hence $v = v'$. ∎

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if G is a dag, then $\leq_G$ defined by $v \leq_G v'$ if there is a path from v to v', is a partial order.

Corollary

every finite dag G can be topologically $\leq_G$-sorted.

Shortest paths in dags

Lemma

in a finite dag, shortest and longest paths can be computed in $O(n)$

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Proof.

shortest path adapting topological sorting: let G be weighted graph with nodes v, v'.

1. initialise v with distance 0
2. while G is non-empty
   - a) set w to a minimal node having some distance (no edges from other such to w), say d
   - b) if $w = v'$ return d
   - c) for each edge $e : w \rightarrow_k w'$ set the distance d' of w' to $\min(d', d + k)$.
   - d) remove w and all edges from it, from G
3. return ∞

∎

Facts on trees

Lemma

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Proof.

let G be a finite tree having, say, n nodes $\{v_1, \ldots, v_n\}$. Setting $v'_1 = v_1$ and v'_{i+1} to be a common ancestor of v'_i and v_{i+1}, we obtain that v'_n is a common ancestor of all nodes. Therefore, v'_n is the root. ■ ■

Lemma (Characterising forests and rooted trees)

1. *in a forest there is* at most one *path from a node v to a node v'*

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1. For a proof by contradiction, suppose there were two paths from v to v'. If $v = v'$, then one of them would be a cycle, contradicting acyclicity. If $v \neq v'$ let $e \neq f$ be the last edges where the paths differ, starting comparing from v'. By being the last such, e and f must have the same target, contradicting in-degree ≤ 1.

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2. (only–if) By the definition of rooted tree and the previous item.
   (if) Uniqueness of paths entails the multigraph can have neither parallel edges nor cycles, so is a dag. If there were edges $e \neq f$ with the same target v', then there would be distinct paths from v to v' via the respective sources of e and f, which cannot be, so in-degree ≤ 1 and we have a forest. Taking v as root shows the forest is a rooted tree. ■

The number of edges and vertices in a tree

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The number of vertices in a finite tree is the number of edges $+1$

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- R^{-1} relates edges to their targets. It is a function from E to $V - \{v\}$, since any edge, say from v' to v'' is the last edge of the unique path from v to v' to v'', and its target v'' is distinct from the root (otherwise there would be a cycle).

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We obtain R is bijection, hence $V - \{v\}$ and E are equinumerous. ∎

Definition (undirected multigraph)

An undirected multigraph is given by

- a set of nodes or vertices V
- a set of edges E
- a map $r\colon E \to \{\{c, d\} \mid c, d \in V\}$ with $e \mapsto r(e)$, that maps every edge e to a set $r(e)$ having one or two elements, its endpoints.
- e is an edge between, joining or incident on its endpoints

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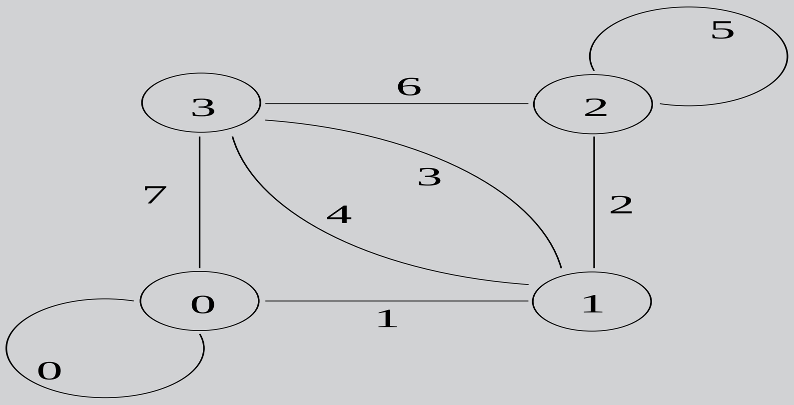
Example

Let $V = \{0,1,2,3\}$, $E = \{0,1,2,\ldots,7\}$ and the function r be defined by

e	$r(e)$	e	$r(e)$
0	$\{0\}$	4	$\{1,3\}$
1	$\{0,1\}$	5	$\{2\}$
2	$\{1,2\}$	6	$\{2,3\}$
3	$\{1,3\}$	7	$\{0,3\}$

Example (Continued)



From directed to undirected multigraphs, and back

Definition

To a directed multigraph an undirected multigraph can be associated by forgetting the directions of edges, defining the set of end-points of an edge e to comprise its source and target: $r(e) = \{src(e), tgt(e)\}$.

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Definition

To an undirected multigraph a directed multigraph can be associated by duplicating each edge e into e_l and e_r directed to the left resp. right, i.e. if $r(e) = \{c,d\}$ then e_l is from d to c, and e_r from c to d.

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Remark

Not inverse to each other, but often preserve properties. For instance, there being a path between two nodes.

Definition

- vertex c is a neighbour of the vertex d, if there exists an edge joining both
- an edge having only one endpoint is a loop
- two edges having the same endpoints are parallel
- the degree of a vertex v is the number of edges having v as endpoint
- a multigraph is vertex- resp. edge-labelled, if there is a function from V resp. E to a set of labels.
- if the labels are numbers, we speak of weights and weighted multigraphs

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Definition (undirected graph)

An undirected graph is an undirected multigraph without parallel edges: then there is for every set of nodes $\{c, d\}$ at most one edge joining c and d.

Definition

- Let $G = (V, E, r)$ be an undirected multigraph
- $G' = (V', E', r')$ is sub-multigraph of G, if $V' \subseteq V$, $E' \subseteq E$ and $r'(e) = r(e)$ for $e \in E'$
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Definition

Let (V, E, r) be an undirected multigraph, and let c, d be vertices

- A tuple $(e_0, e_1, \ldots, e_{\ell-1}) \in E^\ell$ is a path from c to d of length ℓ, if there are vertices $v_0, v_1, \ldots, v_\ell$ with $v_0 = c$, $v_\ell = d$, and $r(e_i) = \{v_i, v_{i+1}\}$ for $i = 0, 1, \ldots, \ell - 1$
- v_0 is the initial or starting node; v_ℓ it its end-node
- $v_1, v_2, \ldots, v_{\ell-1}$ are the intermediate nodes
- For every node $v \in V$, the empty tuple $() \in E^0$ is the empty path with starting node v and end-node v

Definition (Continued)

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from c to e
- A path is closed, if its starting and end-nodes are the same
- A cycle is a non-empty closed simple path

Definition (Continued)

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- Path is simple if non-empty and no repeated nodes (exception $v_0 = v_\ell$), edges.
- For every path $(e_0, e_1, \ldots, e_{\ell-2}, e_{\ell-1})$ from c to d there is the inverse path $(e_{\ell-1}, e_{\ell-2}, \ldots, e_1, e_0)$ from d to c
- The concatenation or composition of the paths $(e_0, e_1, \ldots, e_{\ell-1})$ from c to d and $(f_0, f_1, \ldots, f_{m-1})$ from d to e is the path

$$(e_0, e_1, \ldots, e_{\ell-1}, f_0, f_1, \ldots, f_{m-1})$$

  from c to e
- A path is closed, if its starting and end-nodes are the same
- A cycle is a non-empty closed simple path
- undirected multigraphs without cycles are called cycle-free

Undirected forests and trees

Definition

- A forest is a cycle-free undirected multigraph
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- leaves are nodes with degree ≤ 1 in a forest

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- In the multigraph in the first example there are the following paths from node 0 to node 3

$$(1,2,6), (1,2,5,6), (1,3), (1,4), (1,3,7,1,3), (1,4,7,1,3), (7)$$

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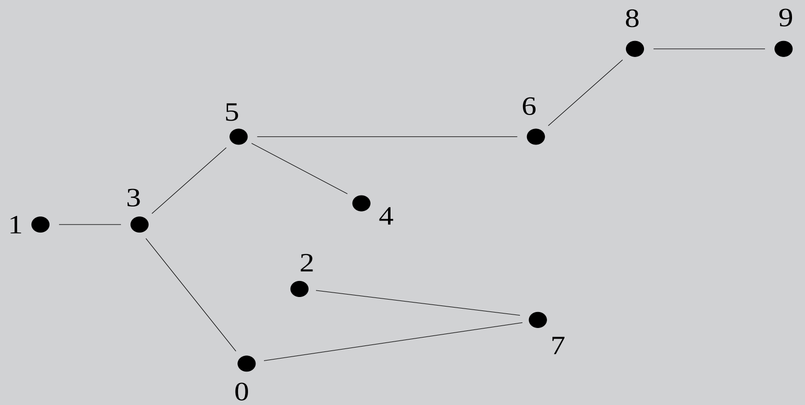
- The multigraph is connected
- There are simple cycles with starting-node 0

$$(0), (1, 2, 6, 7, (1, 3, 7), (1, 4, 7), (7, 3, 1), (7, 4, 1), (7, 6, 2, 1)$$

Example

The following graph is a connected forest, and therefore a tree; its leaves are $1, 2, 4, 9$



Characterising undirected trees

Lemma

For an undirected multigraph G, the following are equivalent.

0) *G is connected but removing any edge makes the graph disconnected*

1) *in G there is a unique simple path between any two nodes*

2) *G is connected and acyclic but adding any edge makes the graph contain a cycle*

Proof.

0)⇒1) Suppose there were two paths between two nodes. W.l.o.g. we may assume these are of minimal (total) length. Then they do not have edges in common, so removing any edge on them the graph would remain connected. Contradiction ■

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Proof.

1)⇒2) By assumption, there is a unique path p between v and v'. Adding a fresh edge e between them, makes the concatenation of p and e into a cycle. ■

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Proof.

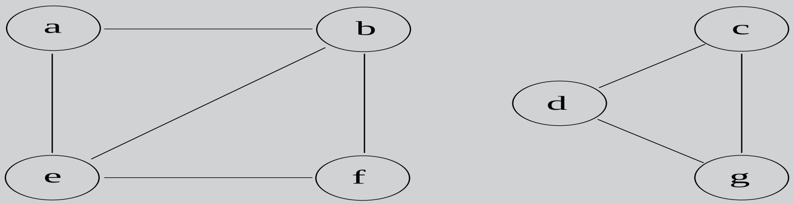
2)⇒0) If removing an edge e between v and v' from G would not affect being connected, there would be a path between v and v' in which e does not occur. But then the concatenation of e and p would be a cycle in G already. Contradiction. ■

Definition

- Let G be an undirected multigraph
- A sub-graph G' of G is a spanning forest of G, if
  1. G is a forest, and
  2. the partitionings of G resp. G' into connected components are the same.
- Then $V' = V$

Example

The following graph has $8 \cdot 3 = 24$ spanning forests



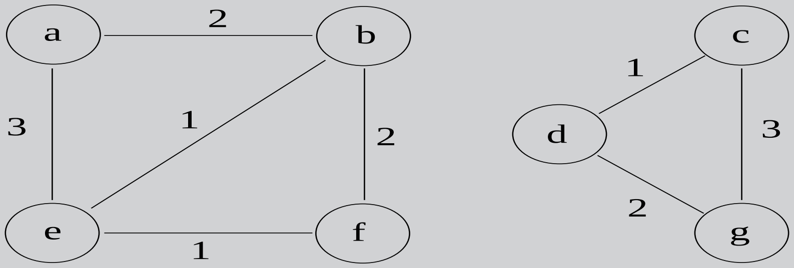
Theorem (Kruskal's algorithm)

1. *Let $G = (V, E, r)$ be an undirected multigraph with weights b*
2. *We want to construct a partitioning of V into connected components, and a set of edges F that constitutes a spanning forest of G having minimal weight $\sum_{e \in F} b(e)$*
3. *We preprocess G by removing all loops and all parallel edges except for a single one of least weight*
4. *The algorithm then proceeds as follows, with complexity $O(\#(V) \cdot \#(E))$*

Set $F = \emptyset$ and $P = \{\{v\} \mid v \in V\}$
For i from 0 to $m - 1$ repeat:
if the nodes v and u of e_i are in distinct blocks of P,
combine both blocks of P and adjoin e_i to F

Example

For the weighted graph



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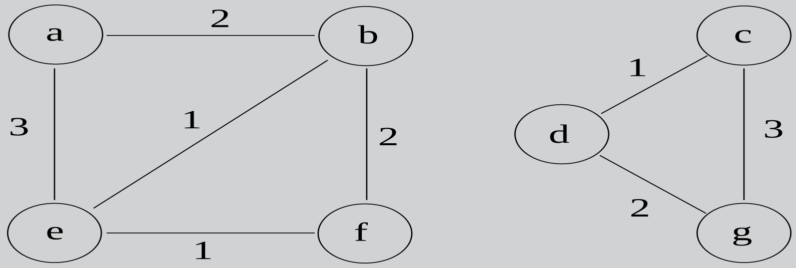

Kruskal's algorithm starts with $F = \emptyset$; $P = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}\}$ and terminates with

$$F = \{\{a, b\}, \{b, e\}, \{c, d\}, \{d, g\}, \{e, f\}\}$$
$$P = \{\{a, b, e, f\}, \{c, d, g\}\}$$

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- We show that the greedy strategy employed, yields a spanning forest of minimal weight

Proof (continued).

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- Since F' has a minimal weight, so has F ∎