Summary last week

- dags as directed acyclic graphs
- topological $\leq$-sorting $(a_0, \ldots, a_n)$ of partial order $\leq$ on $\{a_0, \ldots, a_n\}$: $i < j$ if $a_i < a_j$.
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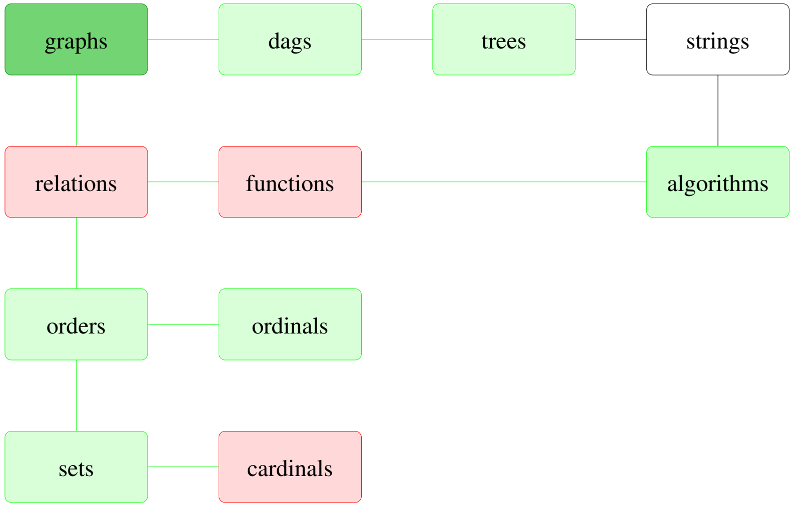
- rooted trees as trees having a root (ancestor of all nodes)
- for trees, number of vertices $=$ number of edges $+1$
- undirected graphs; edges have set of endpoints $\{u, v\}$ (instead of source,target)
- undirected versions of directed notions: path, cycles, forest, tree, ...
- spanning tree of graph as tree having same connected components
- Kruskal's spanning tree algorithm by adjoining edges of least weight (greedy)

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Reminder: cardinals

Motivation/intuition

Capture cardinals as in counting: e.g. 1, 2, 100.
(only number no order)

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Definition

If there exists a bijection $f: M \to N$, then the sets M and N are equinumerous or equipollent. Cardinals represent equinumerous sets.

Example

Each finite set equinumerous to set $\{m \mid m < n\}$ for some $n \in \mathbb{N}$.

Example

$\mathbb{N} \cup \{*\}$ is equinumerous to $\mathbb{N}$; witnessed by bijection f mapping $*$ to 0, and n to $n + 1$.

Definition

- Set A is finite if there exist $n \in \mathbb{N}$ and bijective function $e: \{0, 1, \ldots, n-1\} \to A$
- then n is unique, denoted by $\#(A) := n$, and called the number or cardinality of A
- the function e is in general not unique, and is called an enumeration of A
- a bijection $\nu: A \to \{0, 1, \ldots, m-1\}$ is called a numbering of A
- an inverse of an enumeration is a numbering and vice versa
- A is infinite if it is not finite, and then we write $\#(A) = \infty$

Cardinalities for operations on finite sets

Lemma

Let $e : \{0, \dots, m-1\} \to A$ *and* $f : \{0, \dots, n-1\} \to B$ *be enumerations of A,B.*

1. $\#(\emptyset) = 0$

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5. $\#(A^B) = \#(A)^{\#(B)} = m^n$, *for* A^B *the set of functions from B to A*

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4. mapping k to $e(k)$ if $k < m$ and to $f(k - m)$ otherwise, is a bijection from $\{0, \ldots, m + n - 1\}$ to $A \cup B$, with inverse numbering given by $c \mapsto e^{-1}(c)$ if $c \in A$ and $c \mapsto f^{-1}(c) + m$ if $c \in B$.

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5. writing $k \in \{0, \ldots, m^n - 1\}$ as $k_{n-1} \ldots k_0$ in base-m, mapping it to the function $g : B \to A$ that maps for $0 \leq i < n$, $f(i)$ to $e(k_i)$ is a bijection to A^B, with inverse numbering of elements of A^B given by mapping a function $g : B \to A$ to the number $\sum_{b \in B} f^{-1}(g(b)) m^{e^{-1}(b)}$ in $\{0, \ldots, m^n - 1\}$.

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   Writing $B = \{b_0, \ldots, b_{n-1}\}$, then $g \colon B \to A$ is uniquely determined by the tuple $(g(b_i))_{i=0}^{n-1}$ in B^m. ∎

Derived cardinalities for operations, inclusion/exclusion

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$$\#(\bigcup_{i=1}^{k} A_k) = \#(A_1 \cup A_2 \cup \ldots \cup A_k) = \#(A_1) + \#(A_2) + \ldots + \#(A_k) = \sum_{i=1}^{k} \#(A_i)\,.$$

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3. *For finite sets A and B,*

$$\#(A - B) = \#(A \setminus B) = \#(A) - \#(A \cap B)\,.$$

Proof.

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(3) Because we have for arbitrary sets that

$$A = (A \setminus B) \cup (A \cap B)$$

with the union disjoint, it follows by (2) that

$$\#(A \setminus B) = \#(A) - \#(A \cap B)$$

Proof.

(2) Given bijections

$$e_1\colon \{0,1,\ldots,m_1-1\} \to M_1, \ldots, e_k\colon \{0,1,\ldots,m_k-1\} \to M_k$$

their composition $e\colon \{0,1,\ldots,m_1+\ldots+m_k-1\} \to M_1 \cup \ldots \cup M_k$ is again a bijection

$$i \mapsto \begin{cases} e_1(i) & i \in \{0,1,\cdots,m_1-1\} \\ e_2(i-m_1) & i \in \{m_1,\cdots,m_1+m_2-1\} \\ \vdots & \vdots \\ e_k(i-m_1-\ldots-m_{k-1}) & i \in \{m_1+\ldots+m_{k-1},\cdots,m_1+\ldots+m_k-1\} \end{cases}$$

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For finite sets $A_1, A_2, \ldots, A_k$

$$\#(\bigcup_{i=1}^{k} A_i) =$$

In particular,
$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$

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For finite sets $A_1, A_2, \ldots, A_k$

$$\#(\bigcup_{i=1}^{k} A_i) = \left(\sum_{I \subseteq \{1,\ldots,k\},\ \#(I)\ odd} \#(\bigcap_{i \in I} A_i) \right) - \left(\sum_{I \subseteq \{1,\ldots,k\},\ \#(I)\ even} \#(\bigcap_{i \in I} A_i) \right)$$

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For finite sets $A_1, A_2, \ldots, A_k$

$$\#(\bigcup_{i=1}^{k} A_i) = \sum_{\substack{I \subseteq \{1,2,\ldots,k\} \\ I \neq \emptyset}} (-1)^{\#(I)-1} \#(\bigcap_{i \in I} A_i)$$

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5 *Let $M_1, M_2, \ldots, M_k$ be finite sets. Then cardinality of their Cartesian product, is the product of their cardinalities:*

$$\#(M_1 \times M_2 \times \ldots \times M_k) = \prod_{i=1}^{k} \#(M_i)\,.$$

In particular, $\#(M^k) = \#(M)^k$

Proof.

(4) By induction on k. In case $k=2$, $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$

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For $k > 2$ we have by the IH

$$\#(\bigcup_{i=1}^{k} A_i) = \#((\bigcup_{i=1}^{k-1} A_i) \cup A_k) = \#(\bigcup_{i=1}^{k-1} A_i) + \#(A_k) - \#(\bigcup_{i=1}^{k-1} (A_i \cap A_k)) =$$

$$= \sum_{\substack{I \subseteq \{1,\dots,k-1\} \\ I \neq \emptyset}} (-1)^{\#(I)-1} \#(\bigcap_{i \in I} A_i) + \#(A_k) -$$

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The final equation holds for the three cases (i) $J = I$, (ii) $J = \{k\}$, (iii) $J = I \cup \{k\}$

Proof.

(5) By assumption we have bijections e_i

$$e_1 \colon \{0,1,\dots,m_1-1\} \to M_1, \dots, e_k \colon \{0,1,\dots,m_k-1\} \to M_k$$

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Therefore, $e\colon \{0,1,\ldots,m_1\cdots m_k-1\}\to M_1\times\ldots\times M_k$ with

$$n\mapsto (e_1(n/m_2\cdots m_k),\ldots,e_{k-1}((n/m_k) \bmod m_{k-1}),e_k(n \bmod m_k))$$

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$$\begin{aligned} i_k &= n \bmod m_k \\ i_{k-1} &= (n/m_k) \bmod m_{k-1} \\ &\vdots \\ i_2 &= (n/(m_3\cdots m_k)) \bmod m_2 \\ i_1 &= n/(m_2\cdots m_k) \end{aligned}$$

the number n is obtained by

$$n := i_1\cdot m_2\cdots m_k + i_2\cdot m_3\cdots m_k+\ldots+i_{k-1}\cdot m_k+i_k$$

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Example

In C-programs, the elements of a multi-dimensional array are stored consecutively in memory, where their order is such that „later indices go faster than earlier ones“. For example, the elements of

```
int M[2][3] = {{3,5,-2},{1,0,2}};
```

are arranged in memory as:

M[0][0]	M[0][1]	M[0][2]	M[1][0]	M[1][1]	M[1][2]
3	5	-2	1	0	2

`M`

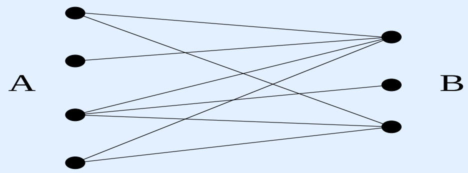
Example (continued)

```
double f(double *z, int m1, int m2, int m3)
{
  ...
}
...
int main( void)
{
   double x, y, A[2][3][4], B[3][4][2];
   ...
   x = f(&A[0][0][0],2,3,4);
   y = f(&B[0][0][0],3,4,2);
   ...
}
```

In the function `f`, the element "‘`z[i][j][k]`"’ can be addressed as `*(z+i*m2*m3+j*m3+k)` the indices `i`, `j`, `k` of the element located at address `z+l` can be computed as `k = l%m3`, `j = (l/m3)%m2` and `i = l/(m2*m3)`

Theorem

6 *Double counting An undirected graph is bipartite, if there exists a partition of its set of nodes in two blocks A and B, such that every edge has one endpoint in A and one in B.*



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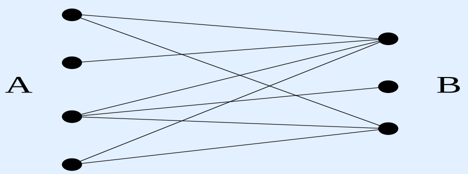

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Proof.

(6) Both sums denote the number of edges ■

Theorem (Pigeon hole principle)

Let $f : M \to N$ *be a function, with M, N finite. If* $\#(M) > \#(N)$*, then there is at least on element* $y \in N$ *having an inverse image with more than one element.*

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Proof.

Assuming the inverse image of each element of N has at most one element, f is injective, and therefore $M \to f(M)$ bijective. Hence $\#(M) = \#(f(M))$ and by $f(M) \subseteq N$ we have $\#(M) \leqslant \#(N)$ ∎

Lemma

Maximum $\geq$ average. For $R = (r_i)_{i \in I}$ a collection of numbers, $\max(R) \geq \frac{\sum R}{\#(I)}$.

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Maximum $\geq$ average. For $R = (r_i)_{i \in I}$ a collection of numbers, $\max(R) \geq \frac{\sum R}{\#(I)}$.

Alternative proof of PHP

Let $R = (\#(f^{-1}(n)))_{n \in N}$. By the lemma $\max(R) \geq \frac{\sum R}{\#(N)} = \frac{\#(M)}{\#(N)} > 1$.

Counting the number of injective functions

Theorem

Let K and M be finite sets having k resp. m elements. Then there are exactly

$$(m)_k := \begin{cases} m(m-1)(m-2)\cdots(m-k+1) & \text{if } k \geqslant 1 \\ 1 & \text{if } k = 0 \end{cases}$$

injective functions from K to M. The number $(m)_k$ is the falling factorial of m and k.

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Theorem

Let K and M be finite sets having k resp. m elements. Then there are exactly

$$(m)_k := \begin{cases} m(m-1)(m-2)\cdots(m-k+1) & \text{if } k \geqslant 1 \\ 1 & \text{if } k = 0 \end{cases}$$

injective functions from K to M. The number $(m)_k$ is the falling factorial of m and k.

Example

Obviously, there are no (total) injective functions from $\{0, 1, 2, 3\}$ to $\{0, 1\}$, which agrees with the theorem as $(2)_4 = 2 \cdot 1 \cdot 0 \cdot -1 = 0$.

Proof.

We show the claim by mathematical induction on k. In the base case, $k = 0$, we have that K is the empty set and the empty function is the only injective function. In the step case, we write

$$K = \{x_0, x_1, \dots, x_k\}$$

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and consider how to construct injective functions $f\colon K \to M$. For x_0 we have m ways to choose an image $f(x_0) \in M$. That element

$$y_0 := f(x_0)$$

then cannot by chosen as image of the other elements of K. That is, as images of $x_1, \dots, x_k$ we must choose elements among $M \setminus \{y_0\}$.

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Counting the number of bijective functions

Theorem

Let K and M be finite sets having m elements each. Then there are exactly

$$m! \ := \begin{cases} m(m-1)(m-2)\cdots 3\cdot 2\cdot 1 & m \geqslant 1 \\ 1 & m = 0 \end{cases}$$

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Proof.

We take some arbitrary but fixed enumeration $e\colon \{0,1,\ldots,m-1\} \to M$. The following function then is a bijection:

$$F\colon \mathcal{P}(M) \to \{0,1\}^m \;,\; T \mapsto (t_0,\ldots,t_{m-1}) \;,\; t_i := \begin{cases} 1 & \text{if } e(i) \in T \\ 0 & \text{otherwise.} \end{cases}$$

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Naming

For $T \subseteq M$, the function $\chi_T : M \to \{0,1\}$ defined by $\chi_T(t) = 1$ if $t \in T$ and 0 otherwise, is the characteristic function of T.

Counting the number of subsets of given size

Theorem

Let M be a finite set with m elements, and let k be a natural number. Then

$$\#(\mathcal{P}_k(M)) = \binom{m}{k} .$$

where $\mathcal{P}_k(M)$ denotes the subsets of size k, and where the binomial coefficient „m choose k“ or „m over k“ is defined by

$$\binom{m}{k} := \frac{m\cdot(m-1)\cdots(m-k+1)}{k\cdot(k-1)\cdots 1} = \begin{cases} \dfrac{m!}{k!(m-k)!} & \text{if } k \leqslant m \\ 0 & \text{otherwise} \end{cases}$$

Proof.

An enumeration $e\colon \{0,1,\ldots,k-1\} \to T$ of a subset T of M having k elements, is obtained by choosing

- an arbitrary element $e(0) \in M$,
- an arbitrary element $e(1) \in M \setminus \{e(0)\}$,
- an arbitrary element $e(2) \in M \setminus \{e(0), e(1)\}$, etc.

Since the order of choosing the elements of T is irrelevant, the number of such choices is

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Infinite counting

Definition

A set M is countably infinite, if there is a bijection

$$e\colon\ \mathbb{N} \to M\ ,\ i \mapsto x_i\,,$$

between M and the set of natural numbers $\mathbb{N}$. M may than be written as

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Example

- The set $\mathbb{N}$ of natural numbers is countably infinite
- And so is the set $\mathbb{Z}$ of integers

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Instead of an enumeration $e\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, we give a numbering $\nu\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We lay-out the pairs (m, n) two-dimensionally

$$\begin{array}{ccccc} (0,0) & (1,0) & (2,0) & (3,0) & \dots \\ (0,1) & (1,1) & (2,1) & (3,1) & \dots \\ (0,2) & (1,2) & (2,2) & (3,2) & \dots \\ (0,3) & (1,3) & (2,3) & (3,3) & \dots \\ \vdots & & & & \end{array}$$

and number diagonally, where we assign to the pair (m, n) the number $\left(\sum_{i=0}^{m+n-1}(i+1)\right) + m$. The function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(m, n) \mapsto \frac{(m+n)(m+n+1)}{2} + m$ is bijective.

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Beyond countably infinite?

Question

From the previous slide we know that products of countably infinite sets are countably infinite again. We can contrast this to that the product of two sets having, say, 4 elements has more than 4 elements (namely $4 \cdot 4 = 16$). Can you find an operation on sets, such that applying it to countably infinite sets yields a set having more than countably infinite elements?