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- pigeon hole principle: $\max(R) \geq \frac{\sum R}{\#(I)}$ for $R = (r_i)_{i\in I}$ collection of numbers
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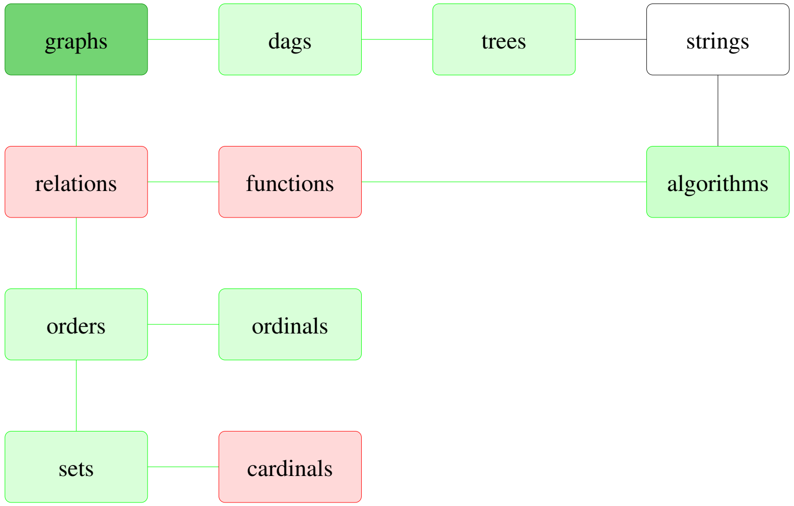
- $\#(A \times B) = \#(A) \cdot \#(B)$, if $A \cap B = \emptyset$ then $\#(A \cup B) = \#(A) + \#(B)$
- $\#(A - B) = \#(A) - \#(A \cap B)$, $\#(A^B) = \#(A)^{\#(B)}$ functions $B \to A$
- $(\#A)_{\#B}$ injective functions $B \to A$; falling factorial
- if $\#A = \#B$, then $\#A!$ bijective functions $B \to A$; if $B = A$, then permutations
- subsets of B, $\#(\mathcal{P}(B)) = 2^{\#B} = \#(\{0,1\}^B)$, characteristic functions $B \to \{0,1\}$
- $\#(\mathcal{P}_k(B)) = \binom{\#B}{k}$ subsets of size k; binomial coefficent $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Infinite counting

Definition

A set A is countably infinite, if there is a bijection

$$e\colon \mathbb{N} \to A\ ,\ i \mapsto a_i\,,$$

between the set of natural numbers $\mathbb{N}$ and A. A may than be written as

$$A = \{a_0, a_1, a_2, \ldots\}\,,$$

e is called an enumeration of A, and e^{-1} a numbering of A.

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Example

- The set $\mathbb{N}$ of natural numbers is countably infinite
- And so is the set $\mathbb{Z}$ of integers

Theorem

The set $\mathbb{N} \times \mathbb{N}$ is countably infinite.

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Proof.

We lay-out the pairs (m, n) two-dimensionally

$$\begin{array}{ccccc} (0,0) & (1,0) & (2,0) & (3,0) & \dots \\ (0,1) & (1,1) & (2,1) & (3,1) & \dots \\ (0,2) & (1,2) & (2,2) & (3,2) & \dots \\ (0,3) & (1,3) & (2,3) & (3,3) & \dots \\ \vdots & & & & \end{array}$$

Instead of an enumeration $e\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, we give a numbering $\nu\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. We number by dove tailing: $(0,0) \mapsto 0$, $(0,1) \mapsto 1$, $(1,0) \mapsto 2$, $(0,2) \mapsto 3$, $(1,1) \mapsto 4$, $(2,0) \mapsto 5$, $(0,3) \mapsto$?

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$(m, n) \mapsto \frac{(m+n)(m+n+1)}{2} + m$; is bijective ∎

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Example

The monoid of words Σ^* is countable, if Σ is a finite alphabet

$$\Sigma^* := \bigcup_{n \geqslant 0} \Sigma^n = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots$$

Beyond countable?

Question

From the previous slides we know that being countable is preserved by various operations (product, subset, image, sequence).

1. Contrast this to that the product of two sets having, say, 4 elements has more than 4 elements (namely $4 \cdot 4 = 16$).
2. Can you find any operation on sets, such that applying it to countable sets yields a set having more than countably many elements?

Theorem (Cantor diagonalisation)

Let Σ be an alphabet containing at least two letters, say a and b, and let $s_0, s_1, s_2, \ldots$ be an infinite sequence of infinite sequences in Σ:

$$\begin{aligned} s_0 &= s_{00}s_{01}s_{02}\ldots \\ s_1 &= s_{10}s_{11}s_{12}\ldots \\ s_2 &= s_{20}s_{21}s_{22}\ldots \\ &\vdots \end{aligned}$$

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Then the sequence

$$d_n := \overline{s_{nn}} := \begin{cases} b & \text{if } s_{nn} = a \\ a & \text{if } s_{nn} \neq a \end{cases}$$

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Proof.

If d were not a new sequence, then there would be an index n such that $d = s_n$, and in particular $d_n = s_{nn}$, contradicting the construction of d as distinct at the diagonal. ∎

Diagonalisation consequences

Corollary

none of the following are countable

1. *the set of infinite sequences over $\{a, b\}$*
2. *functions $2^{\mathbb{N}}$; as infinite sequence is function $\mathbb{N} \to 2 = \{a, b\}$*
3. *subsets $\mathcal{P}(\mathbb{N})$; by characteristic function $2^{\mathbb{N}}$*
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Answer

Via injective functions.

Comparing set sizes

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For sets A and B, we write $|A| \leq |B|$, if there is an injective function $f : A \to B$.

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Suspicion for 3rd item

there is a bijection between A and B

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Let $f\colon A \to B$ and $g\colon B \to A$ be injective functions. Then there is a bijection $f' : A \to B$

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Example (Picture on the board/animation next slide)

Let $A = \mathbb{N}$, $B = \{a\}^*$, and $f\colon A \to B$, $g\colon B \to A$ be defined by:

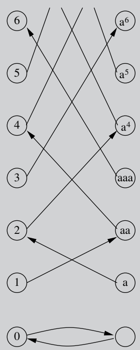
$$f(n) := a^{2n}$$
$$g(a^n) := 2n$$

f and g are injective; a bijection $f'\colon A \to B$ can be constructed from f,g by:

$$f'(n) := \begin{cases} \epsilon & \text{if } n = 0 \\ g^{-1}(n) = a^{\frac{n}{2}} & n \text{ has odd number of 2-factors} \\ f(n) = a^{2n} & \text{otherwise} \end{cases}$$

Animation of construction of bijection f' from injections f, g

Example (Continued)



graph of f union g

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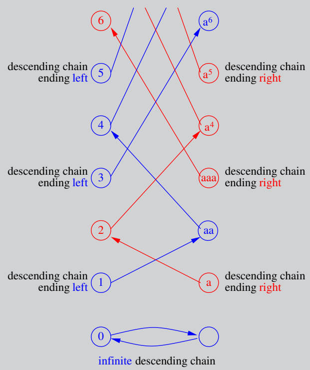

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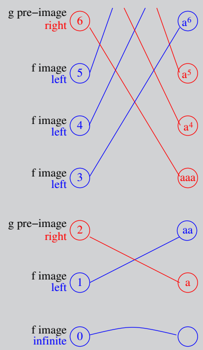

bijection f' by choosing g pre−image for nodes on chains ending right and f image for other nodes

Proof of Schröder–Bernstein theorem

Proof.

We construct $f' : A \to B$ and $g' : B \to A$ inverse to each other, as in animation.

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- define $f'(a)$ for $a \in A$ by cases on the colour of a:
  - **a)** $f'(a) := g^{-1}(a)$ (g pre-image if a is red; pre-image exists as a-chain ends on right)
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  - **b)** $g'(b) := g(b)$ (otherwise g image)
- verify f', g' inverse to each other. $f' ; g'$ ($g' ; f'$ analogous) by cases on colour $a \in A$:
  - **a)** $g'(f'(a)) = g'(g^{-1}(a)) = g(g^{-1}(a)) = a$, as $g^{-1}(a)$ is red if a is, being on same chain.
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Partially ordering sets up to equinumerosity

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for bijections $g : A' \to A$ and $g' : B \to B'$, composition $g ; f ; g' : A' \to B'$ is injection. ∎

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$\leq$ is a partial order on the collections $|M|$

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$|\mathbb{N}| < |\mathbb{R}|$

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Remark

An equivalence class contains all objects having the same property

Example

- Relating equinumerous sets is an equivalence (same $\#$)
- Relating n to m if $n \pmod{k} = m \pmod{k}$ is an equivalence
- Relating $\frac{n}{m}$ and $\frac{n'}{m'}$ if $n \cdot m' = m \cdot n'$ is an equivalence (same normalised fraction)

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$$\begin{aligned}\sim = \{&(000,000),(001,001),(001,010),(001,100),(010,001),\\ &(010,010),(010,100),(100,001),(100,010),(100,100),\\ &(011,011),(011,101),(011,110),(101,011),(101,101),\\ &(101,110),(110,011),(110,101),(110,110),(111,111)\}\end{aligned}$$

That is, 000, 001 $\sim$ 010 $\sim$ 100, 011 $\sim$ 101 $\sim$ 110, 111

Equivalence classes: $\{000\},\{001,010,100\},\{011,101,110\},\{111\}$ (prop: same # of 1s)

Example

- Relating equinumerous sets is an equivalence (same #)
- Relating n to m if $n \pmod k = m \pmod k$ is an equivalence
- Relating $\frac{n}{m}$ and $\frac{n'}{m'}$ if $n \cdot m' = m \cdot n'$ is an equivalence (same normalised fraction)

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System of representatives: $\{000,001,011,111\},\{000,010,011,111\},\ldots$

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$x \sim z \Leftrightarrow [x] = [z]$ *for equivalence relation* $\sim$

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Lemma

Let $f : M \to N$ *be a function. Then*

$$x \sim z \;:\Leftrightarrow f(x) = f(z)$$

defines an equivalence relation. The equivalence classes are the inverse images $f^{-1}(y) = \{x \in M \mid f(x) = y\}$ *for* $y \in f(M)$.

Definition

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(1) *Let P be a partition of M. Then is $\sim$ is an equivalence relation on M, such that*

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(3) *The functions $P \mapsto \sim$ in (1) and $\sim \mapsto P$ in (2) are inverse to each other*

From orders to equivalence relations

Lemma

if $\leq$ is a reflexive, transitive, then $\leq \cap \geq$ is induced equivalence relation.

Proof.

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Example

1. $\frac{n}{m} \leq \frac{n'}{m'}$ if $n \cdot m' \leq m \cdot n'$ induces the equivalence on (positive) fractions above
2. relating sets by injections induces equinumerosity
3. $\leq$ on natural numbers induces equality $=$

Elementary number theory: Euclid

Definition

- $d \in \mathbb{Z}$ is a divisor of $a \in \mathbb{Z}$, if there exists a $c \in \mathbb{Z}$ such that $a = c \cdot d$
- „d divides a“, „a is a multiple of d“ $d \mid a$
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Definition

Let $a, b \in \mathbb{Z}$, $a, b \neq 0$

- The greatest common divisor $\gcd(a, b)$ of a and b divides a and b, and for all c such that $c \mid a$ and $c \mid b$, c divides $\gcd(a, b)$
- The least common multiple $\text{lcm}(a, b)$ of a and b is a multiple of both a and b, and for all c such that $a \mid c$ and $b \mid c$, c is a multiple of $\text{lcm}(a, b)$

Theorem

Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$, $b \neq 0$ and $a \neq c \cdot b$; then

$$\gcd(a, b) = \gcd(|a|, |b|) \quad \textit{and} \quad \gcd(a, b) = \gcd(a - c \cdot b, b)$$

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- If an integer d divides a and b, then it also divides $a - c \cdot b$. Vice versa, if d divides $a - c \cdot b$ and b, then it also divides a.
- the common divisors of a and b are the common divisors of $a - c \cdot b$ and b, and therefore they have the same greatest common divisors as well

∎

Theorem (Euclidean algorithm for integers)

The greatest common divisor of non-zero integers can be computed as follows:

> *Replace the integers by their absolute values.*
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> *While the integers are distinct, repeat:*
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> *Replace the larger of the two by the difference of the larger and the smaller.*
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Theorem (Variant)

Replace the integers by their absolute values.
While neither integer is a multiple of the other, repeat:
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Example

We have $\gcd(138, -48) = 6$, according to the first method:

$$\begin{aligned}\gcd(138,-48) &= \gcd(138,48) = \gcd(90,48) = \gcd(42,48)\\ &= \gcd(42,6) = \gcd(36,6) = \gcd(30,6)\\ &= \gcd(24,6) = \gcd(18,6) = \gcd(12,6)\\ &= \gcd(6,6) = 6\end{aligned}$$

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The second method yields

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Theorem (Bézout's lemma)

Let a and b be non-zero integers. Then there exist natural numbers u and v with

$$u \cdot a + v \cdot b = \gcd(a,b)$$

which can be computed by the following algorithm

Set $A = (|a|, 1, 0)$ and $B = (|b|, 0, 1)$.
While B_1 does not divide A_1, do:
 Compute the integer quotient of A_1 and B_1.
 Set $C = B$.
 Set $B = A - q \cdot C$ (componentwise)
 Set $A = C$.
Set $u = sgn(a) \cdot B_2$ and $v = sgn(b) \cdot B_3$.

Proof.

- Let $T = (T_1, T_2, T_3)$ be a triple of integers and $(*)$ the property

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- The first two triples in the algorithm have this property, hence all the subsequent triples have it as well. Restricting to the first components of triples the Euclidean algorithm is obtained. Therefore, we have for the final triples B

$$\gcd(a, b) = B_1 = |a| \cdot B_2 + |b| \cdot B_3 = \underbrace{(\operatorname{sgn}(a) \cdot B_2)}_{u} \cdot a + \underbrace{(\operatorname{sgn}(b) \cdot B_3)}_{v} \cdot b$$

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A	B	q
$(138, 1, 0)$	$(48, 0, 1)$	2
$(48, 0, 1)$	$(42, 1, -2)$	1
$(42, 1, -2)$	$(6, -1, 3)$	

Theorem (Computing the least common multiple)

Let a and b be non-zero integers. Then

$$\operatorname{lcm}(a, b) = \frac{|a| \cdot |b|}{\gcd(a, b)} .$$

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Obviously,

$$m := \frac{|b|}{\gcd(a, b)} \cdot |a| = \frac{|a|}{\gcd(a, b)} \cdot |b|$$

is a multiple both of a and b, hence a common multiple. We show that m is the least common multiple of a and b. To that end, let z be an arbitrary positive common multiple of a and b. Then there are integers c, d with

$$z = c \cdot a \quad \text{and} \quad z = d \cdot b$$