Summary last week

- A countably infinite if enumeration $\mathbb{N} \to A$; countable if finite or countably infinite.
- countability preserved by subset, image, union, cartesian product
- non-countability of infinite sequences, $2^{\mathbb{N}}$, $\mathcal{P}(\mathbb{N})$, $\mathbb{R}$ by diagonalisation (Cantor)
- injections $f : A \to B$, $g : B \to A$, then exists bijection $A \to B$ (Schröder–Bernstein)
- collections $|\_|$ of equinumerous sets partially ordered by injections; $\mathbb{N} < \mathbb{R}$.

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- if $\sim$ equivalence on A, then $[a] = \{b \mid a \sim b\}$ is equivalence class of $a \in A$
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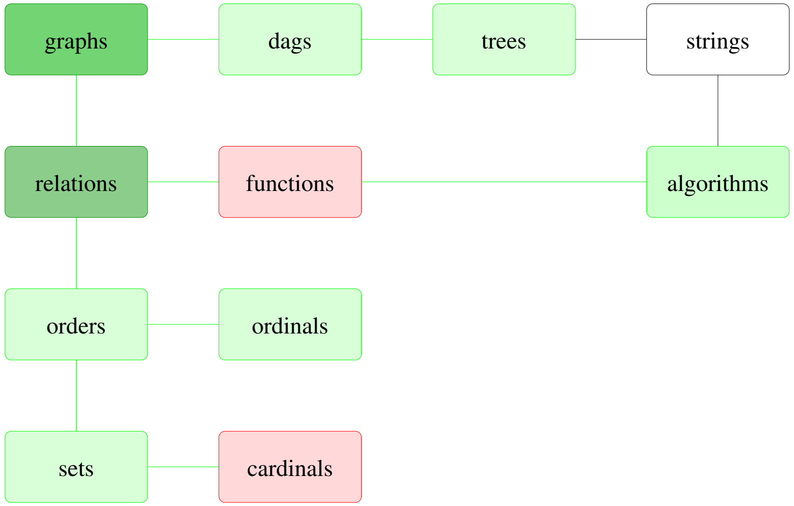
- algorithm for $\text{gcd}(x, y)$ with $x, y \in \mathbb{Z}$ by subtraction, division modulo (Euclid)
- extended algorithm for u, v with $\text{gcd}(x, y) = u \cdot x + v \cdot y$ (Bézout); $\text{lcm}(x, y) = \frac{x \cdot y}{\text{gcd}(x, y)}$

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Discrete structures



Theorem (Bézout's lemma)

for $a, b \in \mathbb{Z}$ not zero, there exist $u, v \in \mathbb{Z}$ with $\gcd(a, b) = u \cdot a + v \cdot b$

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may stop at 1 since 1 is least possible divisor, it's trivial. $u = -7$ and $v = 18$
indeed $1 = \gcd(77,30) = (-7) \cdot 77 + 18 \cdot 30 = -539 + 540$

The divisibility order | (recall from weeks 4 and 5)

Lemma

divisibility $|$ is a well-founded partial order on the positive natural numbers $\mathbb{N}_{>0}$

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note: if $x \mid y$ then $x + \ldots + x = y$ hence $x \leq y$ (for y positive)

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$\Rightarrow$ proofs by well-founded induction on | for statements on $\mathbb{N}_{>0}$ and $\mathbb{N}_{>1} = \mathbb{N} - \{0, 1\}$

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- p is prime if $p \in \mathbb{N}_{>1}$ and for all x, y, if $p \mid x \cdot y$ then $p \mid x$ or $p \mid y$
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- Assume p indecomposable and suppose $x \mid p$ with $x \in \mathbb{N}_{>1}$, i.e. $x \cdot y = p$ for some y. By p being indecomposable, then x, y are trivial, so $p = x$ and p is $\mid$-minimal

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- Assume p $|$-minimal and suppose $p \mid x \cdot y$, i.e. $p \cdot d = x \cdot y$ for some d. Either $p \mid x$ or else $\gcd(p, x) = 1$ by p being $|$-minimal. Then $1 = u \cdot p + v \cdot x$ for some u, v (Bézout): $y = y \cdot 1 = y \cdot (u \cdot p + v \cdot x) = y \cdot u \cdot p + y \cdot v \cdot x = y \cdot u \cdot p + v \cdot p \cdot d = (y \cdot u + v \cdot d) \cdot p$ hence $p \mid y$. That is, either $p \mid x$ or $p \mid y$, so p is prime ∎

Theorem (Fundamental theorem of arithmetic, FTA)

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  recall from week 5.

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  Suppose $i \in I$. Then $p_i \mid \prod p_I = \prod q_J$, so $\exists j \in J$ such that $p_i \mid q_j$ hence $p_i = q_j$ (by the lemma twice). Therefore, $\prod p_{I-\{i\}} = \frac{\prod p_I}{p_i} = \frac{\prod q_J}{q_j} = \prod q_{J-\{j\}}$, and by the IH $p_{I-\{i\}}$ and $q_{J-\{j\}}$ are the same up to order, hence so are p_I and q_J. ■

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Remark

FTA links numbers wrt addition $(+,-)$ to numbers wrt multiplication $(\cdot, \div)$. Connections between both hard in general, cf. Goldbach's conjecture: if $n > 2$, then $n = p_i + p_j$.

Operations on numbers via exponents of prime factors

Corollary (to FTA)

any $n \in \mathbb{N}_{>0}$ can be uniquely written as $p_k^e := \prod_{i=1}^{k} p_i^{e_i}$ given a long enough initial segment p_k of the prime numbers in ascending order, and collection e_k of exponents

Operations on numbers via exponents of prime factors

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any $n \in \mathbb{N}_{>0}$ can be uniquely written as $p_k^e := \prod_{i=1}^{k} p_i^{e_i}$ given a long enough initial segment p_k of the prime numbers in ascending order, and collection e_k of exponents

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- $77 = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^1 \cdot 11^1$ exponents $e = (0,0,0,1,1)$ and $28 = 2^2 \cdot 3^0 \cdot 5^0 \cdot 7^1 \cdot 11^0$

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$p_n^e \cdot p_n^f = p_n^{e+f}$, $p_n^e \div p_n^f = p_n^{e \dot{-} f}$, $\gcd(p_n^e, p_n^f) = p_n^{\min(e,f)}$, and $\mathrm{lcm}(p_n^e, p_n^f) = p_n^{\max(e,f)}$

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Proof.

writing $|a| = p_n^e$ and $|b| = p_n^f$ for n large enough, by the previous corollary:
$\mathrm{lcm}(a,b) = \mathrm{lcm}(p_n^e, p_n^f) = p_n^{\max(e,f)} = p_n^{e+f-\min(e,f)} = \frac{(p_n^e)\cdot(p_n^f)}{\gcd(p_n^e, p_n^f)} = \frac{|a|\cdot|b|}{\gcd(a,b)}$
using $\max(x,y) = x + y - \min(x,y)$ for natural numbers x, y. ■

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RSA outline, omitting some conditions

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RSA ingredients developed on following slides:

modulo, Euler (RSA case), fast exponentiation, Chinese remainder (speed-up)

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Definition (modulo some positive natural number n)

- integers a, b are congruent modulo n, denoted by $a \equiv b \pmod{n}$ if remainders $a \bmod n$ and $b \bmod n$ after division by n are the same

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Remark

As system of representatives we usually employ the smallest non-negative remainders $\{0, 1, 2, \ldots, n-1\}$ or the absolutely-smallest remainders

$$\begin{cases} \{-n/2 + 1, \ldots, -1, 0, 1, \ldots, n/2\} & \text{if } n \text{ is even} \\ \{-(n-1)/2, \ldots, -1, 0, 1, \ldots, (n-1)/2\} & \text{if } n \text{ is odd.} \end{cases}$$

Modulo (continued)

Example

We have $\mathbb{Z}/5\mathbb{Z} = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4}\} = \{\overline{-2},\overline{-1},\overline{0},\overline{1},\overline{2}\}$; moreover
$\overline{0} = \{0,5,10,15,\dots\} = \overline{5}$, and $\overline{2}+\overline{4} = \overline{6} = \overline{1}$ and $\overline{4}\cdot\overline{4}\cdot\overline{3} = \overline{4\cdot 4}\cdot\overline{3} = \overline{1}\cdot\overline{3} = \overline{3}$.

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The functions

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Example

In many programming languages there is a data type for integers corresponding to $\mathbb{Z}/2^{2^n}\mathbb{Z}$ for some $n \geq 3$. For example `unsigned int` in C corresponds to $n = 5$ resp. $n = 6$. For $n = 5$, i.e. a 32-bits architecture, the sum of $2^{2^5} - 1 = 2^{32} - 1$ and 1 is 0.

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Definition

A congruence class $\overline{a}$ modulo n is invertible, if there is a congruence class $\overline{b}$ modulo n such that $\overline{a}\cdot\overline{b} \equiv \overline{1} \pmod{n}$, i.e. if $a \cdot b - 1 = k \cdot n$ for some k.

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$\overline{a}$ modulo n is invertible for non-zero a iff $\gcd(a, n) = 1$; in that case, we can compute using Bézout's lemma, integers u, v such that $u \cdot a + v \cdot n = 1$ and $\overline{a}^{-1} = \overline{u}$

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Proof.

if $\gcd(a, n) = 1$ and $u \cdot a + v \cdot n = 1$, then $\overline{1} = \overline{u} \cdot \overline{a} + \overline{v} \cdot \overline{n} = \overline{u} \cdot \overline{a}$. vice versa, if $\overline{a}$ invertible, then $\overline{a} \cdot \overline{b} = \overline{1}$ for some b, hence $\overline{a \cdot b - 1} = \overline{0}$; and therefore $n \mid (a \cdot b - 1)$. thus $\gcd(a, n) = 1$, as $\gcd(a, n)$ divides n hence $a \cdot b - 1$, and a hence $a \cdot b$ ∎

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Corollary (cancellation by multiplication with $\overline{a}^{-1}$)

if $0 < a < p$ and $a \cdot b \equiv a \cdot c \pmod{p}$ with p prime, then $b \equiv c \pmod{p}$

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where we use cancellation again to show $\overline{1 \cdot a}, \overline{2 \cdot a}, \ldots, \overline{(p-1) \cdot a}$ are all distinct and also from $\overline{0}$, so that they must be a permutation of the congruence classes $\overline{1}, \overline{2}, \ldots, \overline{(p-1)}$, to conclude their products are the same (double counting). ∎

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Corollary (Euler's theorem, RSA case)

for all primes p, q, and integers a with $\gcd(a, p \cdot q) = 1$, $a^{(p-1)\cdot(q-1)} \equiv 1 \pmod{p \cdot q}$

Theorem (Fermat's little theorem, FLT)

for prime p, and integer a with $p \nmid a$, we have $a^{p-1} \equiv 1 \pmod{p}$

Proof.

by cancellation of $\overline{1 \cdot 2 \cdots (p-1)}$ from

$$\overline{1 \cdot 2 \cdots (p-1)} \cdot \overline{a}^{p-1} = \overline{1 \cdot a} \cdot \overline{2 \cdot a} \cdots \overline{(p-1) \cdot a} = \overline{1 \cdot 2 \cdots (p-1)} \cdot \overline{1}$$

where we use cancellation again to show $\overline{1 \cdot a}, \overline{2 \cdot a}, \ldots, \overline{(p-1) \cdot a}$ are all distinct and also from $\overline{0}$, so that they must be a permutation of the congruence classes $\overline{1}, \overline{2}, \ldots, \overline{(p-1)}$, to conclude their products are the same (double counting). ∎

Corollary (Euler's theorem, RSA case)

for all primes p, q, and integers a with $\gcd(a, p \cdot q) = 1$, $a^{(p-1)\cdot(q-1)} \equiv 1 \pmod{p \cdot q}$

Proof.

By FTA and $p, q \mid a^{(p-1)\cdot(q-1)} - 1$, from FLT twice, with a^{p-1}, q resp. a^{q-1}, p. ∎

Fast exponentiation

Example

We compute: $3^9 = 3^{(1001)_2} = 3^{2^3} \cdot 3^{2^0} = 3^8 \cdot 3^1 = ((3^2)^2)^2 \cdot 3 = 19683$. The computation uses 4 multiplications, of which 3 are for squaring.

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Theorem (exponentiation by squaring)

Let a be an integer and let n be a positive integer with binary representation $b_k b_{k-1} \cdots b_0$ where $b_k = 1$; in symbols $(b_k b_{k-1} \cdots b_0)_2 = n$. We can then compute the power a^n by squaring (and possibly multiplying) k-times:

> *Set $x = a$.*
> *For i from $k-1$ down to 0 repeat:*
> *Set $x = x^2$.*
> *If $b_i = 1$, set $x = x * a$.*

Fast exponentiation (continued)

Proof.

- By mathematical induction on k; for $k = 0$ $n = 1$ and the algorithm yields $a^1 = a$
- For $k > 0$ we write

$$n = \sum_{i=0}^{k} b_i 2^i = m \cdot 2 + b_0 \quad \text{with} \quad m = \sum_{i=1}^{k} b_i 2^{i-1} = \sum_{i=0}^{k-1} b_{i+1} 2^i$$

By the induction hypothesis, the first $k-1$ loops yield the value a^m; therefore, the last time ($i = 0$) yields

$$(a^m)^2 \cdot a^{b_0} = a^n$$

Fast exponentiation (continued)

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∎

Remark

during exponentiation modulo some number n, no numbers $\geq n$ need to be used.

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q) = 1$*, then the following function* crt *from numbers* $0 \leq x < p \cdot q$ *to pairs* (a,b) *with* $0 \leq a < p$ *and* $0 \leq b < q$*, is a* *bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Chinese remainder theorem, bijection

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$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0				
1					
2					

$$0 \mapsto (0,0)$$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

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0	0				
1		1			
2					

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Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0				
1		1			
2			2		

$$2 \mapsto (2,2)$$

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$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0			3	
1		1			
2			2		

$3 \mapsto (0, 3)$

Chinese remainder theorem, bijection

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$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0			3	
1		1			4
2			2		

$4 \mapsto (1, 4)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p, q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a, b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0			3	
1		1			4
2	5		2		

$5 \mapsto (2, 0)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p, q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a, b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$)

a \ b	0	1	2	3	4
0	0	6		3	
1		1			4
2	5		2		

$6 \mapsto (0, 1)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q)=1$*, then the following function* crt *from numbers* $0 \leq x < p \cdot q$ *to pairs* (a,b) *with* $0 \leq a < p$ *and* $0 \leq b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p=3, q=5$)

a \ b	0	1	2	3	4
0	0	6		3	
1		1	7		4
2	5		2		

$7 \mapsto (1,2)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q)=1$*, then the following function* crt *from numbers* $0 \leq x < p \cdot q$ *to pairs* (a,b) *with* $0 \leq a < p$ *and* $0 \leq b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p=3, q=5$)

a \ b	0	1	2	3	4
0	0	6		3	
1		1	7		4
2	5		2	8	

$8 \mapsto (2,3)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

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$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p=3, q=5$)

a \ b	0	1	2	3	4
0	0	6		3	9
1		1	7		4
2	5		2	8	

$9 \mapsto (0,4)$

Chinese remainder theorem, bijection

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if $\gcd(p,q)=1$*, then the following function* crt *from numbers* $0 \leq x < p \cdot q$ *to pairs* (a,b) *with* $0 \leq a < p$ *and* $0 \leq b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p=3, q=5$)

a \ b	0	1	2	3	4
0	0	6		3	9
1	10	1	7		4
2	5		2	8	

$10 \mapsto (1,0)$

Chinese remainder theorem, bijection

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$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$**)**

a \ b	0	1	2	3	4
0	0	6		3	9
1	10	1	7		4
2	5	11	2	8	

$11 \mapsto (2, 1)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p, q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a, b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$**)**

a \ b	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7		4
2	5	11	2	8	

$12 \mapsto (0, 2)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p, q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a, b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$**)**

a \ b	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	

$13 \mapsto (1, 3)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p, q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a, b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 5$**)**

a \ b	0	1	2	3	4
0	0	6	12	3	9
1	10	1	7	13	4
2	5	11	2	8	14

$14 \mapsto (2, 4)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a,b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 3$)

a \ b	0	1	2
0	0		
1			
2			

$0 \mapsto (0,0)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a,b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 3$)

a \ b	0	1	2
0	0		
1		1	
2			

$1 \mapsto (1,1)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a,b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 3$)

a \ b	0	1	2
0	0		
1		1	
2			2

$2 \mapsto (2,2)$

Chinese remainder theorem, bijection

Theorem (Chinese Remainder, bijection)

if $\gcd(p,q) = 1$*, then the following function* crt *from numbers* $0 \le x < p \cdot q$ *to pairs* (a,b) *with* $0 \le a < p$ *and* $0 \le b < q$*, is a bijection:*

$$x \mapsto (x \bmod p, x \bmod q)$$

Example ($p = 3, q = 3$)

a \ b	0	1	2
0	0		
1		1	
2			2

$3 \mapsto (0,0)$ $\gcd(p,q) = 3 \ne 1$, crt not a bijection

Chinese remainder theorem, bijection

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Proof.

sufficient to prove injectivity.

Chinese remainder theorem, bijection

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Proof.

sufficient to prove injectivity. suppose $0 \le x, x' < p \cdot q$. if $\mathrm{crt}(x) = \mathrm{crt}(x')$, then $x \equiv x' \pmod p$ and $x \equiv x' \pmod q$, hence $p, q \mid x - x'$. Thus

$$p \cdot q = \frac{p \cdot q}{1} = \frac{p \cdot q}{\gcd(p,q)} = \mathrm{lcm}(p,q) \mid x - x'$$

that is, solutions are $p \cdot q$ apart, so $x - x' = 0$ and $x = x'$.

Chinese remainder theorem, bijection

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Theorem (Chinese remainder theorem, Bézout)

Let p *and* q *be positive integers such that* $\gcd(p,q) = 1$*, and let* a *and* b *be arbitrary integers. The congruence system*

$$\begin{aligned} x &\equiv a \pmod p \\ x &\equiv b \pmod q \end{aligned}$$

then has the unique solution $x \equiv vqa + upb \pmod{pq}$ *where the integers* u *and* v *such that* $up + vq = 1$ *can be computed using Bézout's lemma.*

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Proof.

- (existence) we show $x = v \cdot q \cdot a + u \cdot p \cdot b$ for $up + vq = 1$ satisfies equations: $x \equiv v \cdot q \cdot a + u \cdot p \cdot b \equiv v \cdot q \cdot a \equiv (1 - u \cdot p) \cdot a \equiv a - u \cdot p \cdot a \equiv a \pmod{p}$ and similarly for $x \equiv b \pmod{q}$
- (uniqueness)

Theorem (Chinese remainder theorem, Bézout)

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- (uniqueness) as before: if both x, x' are solutions to the two equations, then $p, q \mid (x - x')$, hence $\text{lcm}(p, q) = \frac{p \cdot q}{\gcd(p,q)} = p \cdot q \mid (x - x')$. That is, solutions are $p \cdot q$ apart, hence unique in $\{0, \ldots, p \cdot q - 1\}$.

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- (uniqueness) as before: if both x, x' are solutions to the two equations, then $p, q \mid (x - x')$, hence $\text{lcm}(p, q) = \frac{p \cdot q}{\gcd(p,q)} = p \cdot q \mid (x - x')$. That is, solutions are $p \cdot q$ apart, hence unique in $\{0, \ldots, p \cdot q - 1\}$. ■

Example

The following congruence system has the unique solution $x \equiv 16 \pmod{35}$

$$x \equiv 1 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

Example

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$$x \equiv 1 \pmod{5}$$
$$x \equiv 2 \pmod{7}$$

We compute integers u and v, such that $u \cdot 5 + v \cdot 7 = \gcd(5,7)$.

$A = (5,1,0)$	$B = (7,0,1)$	$q = 0$
$A = (7,0,1)$	$B = (5,1,0)$	$q = 1$
$A = (5,1,0)$	$B = (2,-1,1)$	$q = 2$
$A = (2,-1,-1)$	$B = (1,3,-2)$	$q = 2$

Hence $u = 3$, $v = -2$ and $\gcd(5,7) = 3 \cdot 5 - 2 \cdot 7 = 1$, and therefore

$$\underbrace{-2}_{v} \cdot \underbrace{7}_{q} \cdot \underbrace{1}_{a} + \underbrace{3}_{u} \cdot \underbrace{5}_{p} \cdot \underbrace{2}_{b} = 16$$

By the theorem, the solution $x \equiv 16 \pmod{35}$ is unique

Chinese remainder, RSA

Theorem (Chinese remainder, RSA)

Let $\gcd(p,q) = 1$ *and let* p' *be inverse of* p *modulo* q*, i.e.* $p \cdot p' \equiv 1 \pmod{q}$*. Then*

$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv b \pmod{q} \end{aligned} \iff x \equiv a + p \cdot ((p' \cdot (b-a)) \bmod q) \pmod{p \cdot q}$$

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$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv b \pmod{q} \end{aligned} \iff x \equiv a + p \cdot ((p' \cdot (b-a)) \bmod q) \pmod{p \cdot q}$$

Proof.

$\Leftarrow$ $x \equiv a + p \cdot ((p' \cdot (b-a)) \bmod q) + k \cdot p \cdot q \equiv a \pmod{p}$

$x \equiv a + p \cdot p' \cdot (b-a) + k \cdot p \cdot q \equiv a + b - a \equiv b \pmod{q}$

Chinese remainder, RSA

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Proof.

$\Leftarrow$ $x \equiv a + p \cdot ((p' \cdot (b-a)) \bmod q) + k \cdot p \cdot q \equiv a \pmod{p}$

$x \equiv a + p \cdot p' \cdot (b-a) + k \cdot p \cdot q \equiv a + b - a \equiv b \pmod{q}$

$\Rightarrow$ previous item shows rhs is a solution. now show it is unique modulo $p \cdot q$.
$0 \le x, x' < p \cdot q$ being solutions entails $x \equiv x' \pmod{p}$ and $x \equiv x' \pmod{q}$, hence $p, q \mid x - x'$. Thus, $p \cdot q = \frac{p \cdot q}{\gcd(p,q)} = \mathrm{lcm}(p,q) \mid x - x'$, so $x - x' = 0$ and $x = x'$. ∎

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Let $\gcd(p,q) = 1$ *and let* p' *be inverse of* p *modulo* q*, i.e.* $p \cdot p' \equiv 1 \pmod{q}$*. Then*

$$\begin{array}{rcl} x & \equiv & a \pmod{p} \\ x & \equiv & b \pmod{q} \end{array} \iff x \equiv a + p \cdot ((p' \cdot (b-a)) \bmod q) \pmod{p \cdot q}$$

Example

Let $p = 3$, $q = 5$ (see above). Then $p' = 2$ ($3 \cdot 2 \equiv 1 \pmod{5}$). E.g. for $a = 1$ and $b = 2$, we obtain $x = 1 + 3 \cdot (2 \cdot (2-1) \bmod 5) = 7$, and 7 is indeed the number we find at coordinates $(a,b) = (1,2)$ in the table on slide 17. For another example, at coordinate $(2,1)$ in the table $x = 2 + 3 \cdot (2 \cdot (1-2) \bmod 5) = 2 + 3 \cdot (-2 \bmod 5) = 2 + 3 \cdot 3 = 11$.

Chinese remainder, RSA

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Application to RSA

Speed up computation of $c^d \bmod (p \cdot q)$ for $\gcd(p,q) = 1$?

Chinese remainder, RSA

Theorem (Chinese remainder, RSA)

Let $\gcd(p,q) = 1$ *and let* p' *be inverse of* p *modulo* q*, i.e.* $p \cdot p' \equiv 1 \pmod{q}$*. Then*

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Application to RSA

Speed up computation of $c^d \bmod (p \cdot q)$ for $\gcd(p,q) = 1$?

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Chinese remainder, RSA

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Let $\gcd(p,q) = 1$ *and let* p' *be inverse of* p *modulo* q*, i.e.* $p \cdot p' \equiv 1 \pmod{q}$*. Then*

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$$\begin{aligned} x &\equiv a \pmod{p} \\ x &\equiv b \pmod{q} \end{aligned} \quad \Longleftrightarrow \quad x \equiv a + p \cdot ((p' \cdot (b - a)) \bmod q) \pmod{p \cdot q}$$

Application to RSA

Speed up computation of $c^d \bmod (p \cdot q)$ for $\gcd(p, q) = 1$?

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2. compute $b := c^{d \bmod (q-1)} \bmod q$; by FLT $c^d \equiv b \pmod{q}$
3. compute $m := a + p \cdot ((p' \cdot (b - a)) \bmod q) \bmod (p \cdot q)$; by CRT $m \equiv c^d \pmod{p \cdot q}$.