A splitting integrator for third order problems with transparent boundary conditions

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Introduction

Water waves in shallow water

- **Russell 1834**: I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses [...]
- **Rayleigh and Boussinesq (1870), Korteweg and De Vries (1895)**: theoretical investigations
- **Zabusky and Kruskal (1965)**: solitions
- **Gardner, Greene, Kruskal and Miura (1967)**: inverse scattering transform

Figure. Russell's experiment (1995), Department of Mathematics, Heriot-Watt University, Edinburgh, Scotland

Collision of solitions

`https://www.youtube.com/watch?v=wEbYELtGZwI`

The Korteweg–de Vries (KdV) equation

$$\begin{cases} u_t + 6uu_x + u_{xxx} = 0, \quad t \in [0, T], \quad x \in \mathbb{R}, \\ u(0, x) = u^0(x). \end{cases} \tag{1}$$

The Korteweg–de Vries (KdV) equation

$$\begin{cases} u_t + g(x)u_x + u_{xxx} = 0, \quad t \in [0, T], \quad x \in \mathbb{R}, \\ u(0, x) = u^0(x). \end{cases} \tag{1}$$

Motivation

Equation (2) finds applications in

- simulations of long ocean waves over an uneven bottom,
- propagation of fairly long waves in the shallow water.

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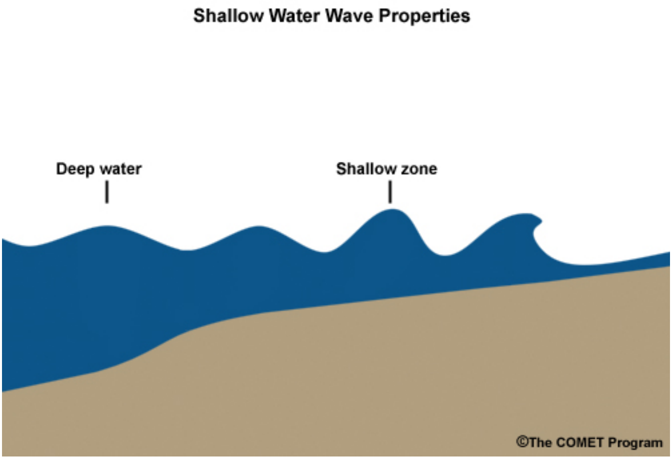
Numerical simulations require

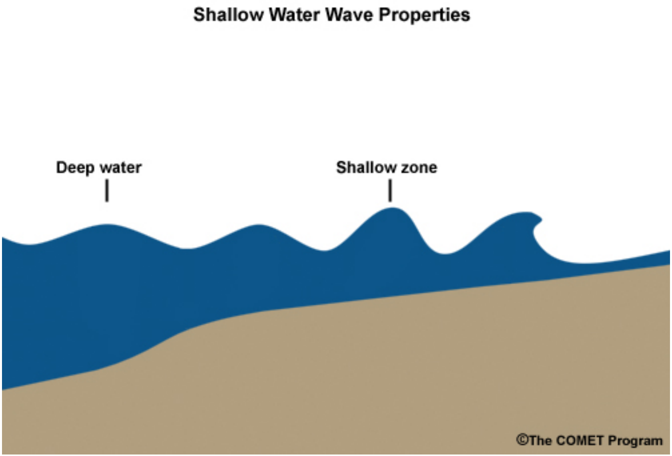
- Temporal discretization
- Spatial discretization

The problem



$$u_t + g(x)u_x + u_{xxx} = 0$$

The problem



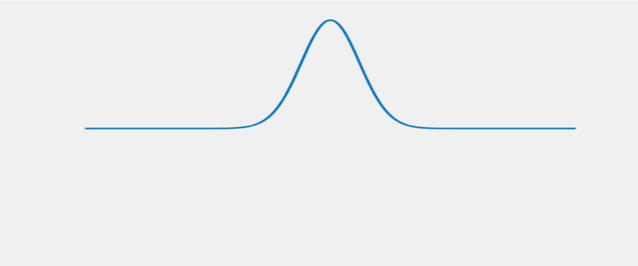
$$u_t + g(x)u_x + u_{xxx} = 0$$

Dispersion

Figure. Drogogne river, France. Picture: https://www.sudouest.fr/

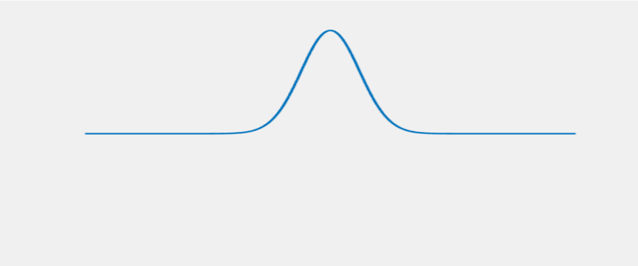
$$u_t + u_{xxx} = 0$$

Dispersion



$$u_t + u_{xxx} = 0$$

Advection



$$u_t + g(x)u_x = 0$$

Temporal discretization

Let $m > 0$ and $\tau = T/m$. Consider the *uniform* time discretization

$$0 = t^0 < t^1 < \cdots < t^m = T, \quad t^k = k\tau.$$

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$$\mathcal{T}^t(v^0) : \begin{cases} v_t + g(x)\, v_x = 0, \\ v(0,x) = v^0(x), \end{cases} \qquad \mathcal{D}^t(w^0) : \begin{cases} w_t + w_{xxx} = 0, \\ w(0,x) = w^0(x). \end{cases}$$

By Lie-Trotter splitting

$$u(\tau, x) \approx u^1(x) = \mathcal{D}^\tau \circ \mathcal{T}^\tau(u^0)$$

Boundary conditions

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Consider the problem

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Boundary conditions

Consider the problem

$$\begin{cases} u_t + g(x)u_x + u_{xxx} = 0, \quad t \in [0, T], \quad {\color{red} x \in (a, b)}, \\ u(0, x) = u^0(x), \\ u(t, a) = \ldots \\ u(t, b) = \ldots \\ u_x(t, b) = \ldots \end{cases}$$

... we must impose boundary conditions on a and b.

A typical choice are periodic boundary conditions.

Periodic boundary conditions

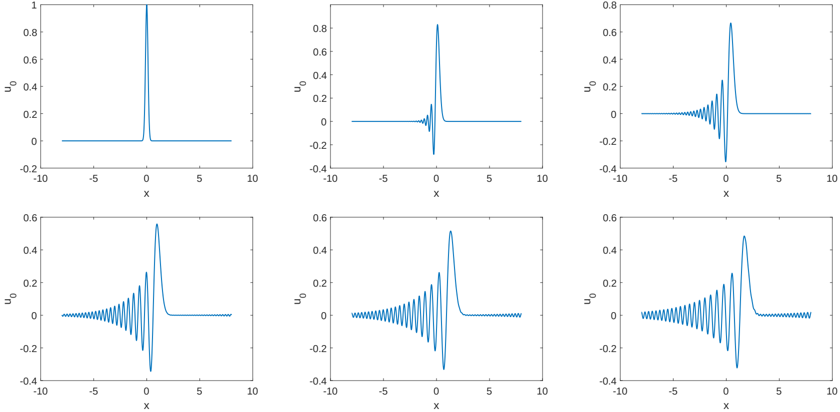


Figure. Time simulation for $T = 2$ with periodic boundary conditions.

Transparent boundary conditions

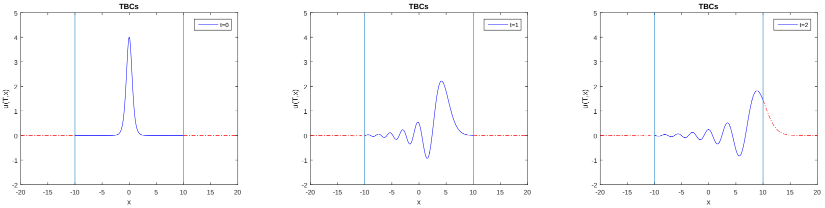


Figure. Time simulation for $T = 2$ with transparent boundary conditions.

Transparent boundary conditions

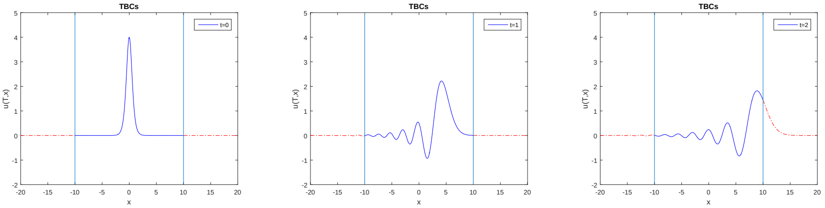


Figure. Time simulation for $T = 2$ with transparent boundary conditions.

No free lunch theorem

- Transparent boundary conditions are difficult to compute
- Transparent boundary conditions are expensive to compute
- Transparent boundary conditions must be carefully design on the underlying numerical scheme

Transparent boundary conditions

We finally obtain the boundary conditions for the interior problem

$$\begin{cases} u^{m+1} + \tau u_{xxx}^{m+1} = u^m - \tau\, g(x) u_x^m, \quad m \geq 0, \quad x \in (a,b), \\ u(0,x) = u^0(x), \\ u^{m+1}(a) - Y_1^0 u_x^{m+1}(a) - Y_2^0 u_{xx}^{m+1}(a) = h_1^m, \\ u^{m+1}(b) - Y_3^0 u_{xx}^{m+1}(b) = h_2^m, \\ u_x^{m+1}(b) - Y_4^0 u_{xx}^{m+1}(b) = h_3^m. \end{cases}$$

where

$$h_1^m = \sum_{k=1}^{m+1} Y_1^k u_x^{m+1-k}(a) + Y_2^k u_{xx}^{m+1-k}(a),$$

$$h_2^m = \sum_{k=1}^{m+1} Y_3^k u_{xx}^{m+1-k}(b), \quad h_3^m = \sum_{k=1}^{m+1} Y_4^k u_{xx}^{m+1-k}(b).$$

Spatial discretization

Given the semi-discrete scheme

$$
\begin{cases}
u^{m+1} + \tau u_{xxx}^{m+1} = u^m - \tau\, g(x) u_x^m \quad m \geq 0, \quad x \in [a,b], \\
u(0,x) = u^0(x), \\
u^{m+1}(a) - Y_1^0 u_x^{m+1}(a) - Y_2^0 u_{xx}^{m+1}(a) = h_1^m, \\
u^{m+1}(b) - Y_3^0 u_{xx}^{m+1}(b) = h_2^m, \\
u_x^{m+1}(b) - Y_4^0 u_{xx}^{m+1}(b) = h_3^m,
\end{cases}
$$

we perform a space discretization by a **pseudo-spectral method**.
In particular, we implement a **dual-Petrov–Galerkin method**.

Numerical results

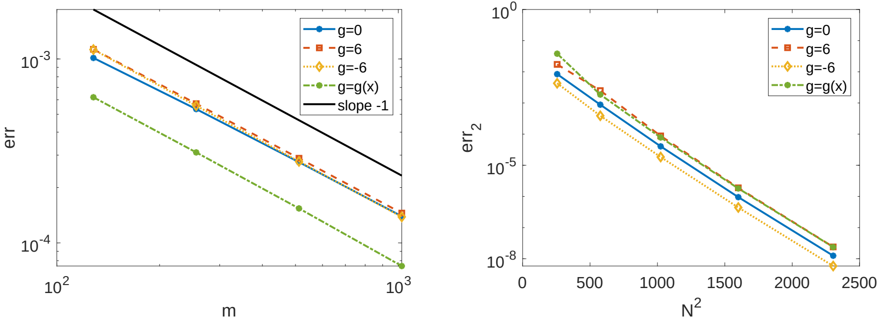
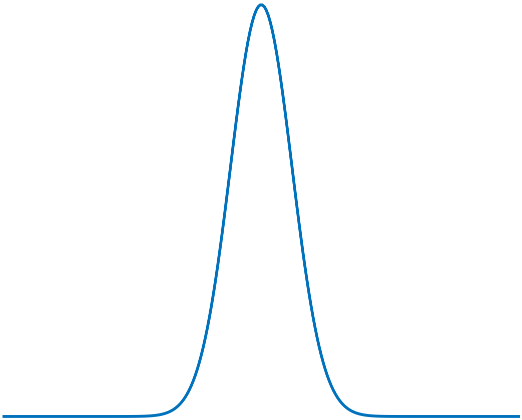


Figure. Left: time convergence, m number of time steps. Right: space convergence, N number of grid points.

Numerical results



Conclusions

- A splitting scheme was used for the time discretization
- Discrete transparent boundary conditions were designed for the particular time scheme
- A pseudo-spectral spatial discretization was presented to achieve fast convergence

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Future work

- Second order time scheme
- Rigorous stability analysis
- 2-D implementation

Thanks for the attention!

L. Einkemmer, A. Ostermann, M. Residori, *A pseudo-spectral splitting method for linear dispersive problems with transparent boundary conditions* (arXiv-2019)