Deep Learning for Regularizing Inverse Problems

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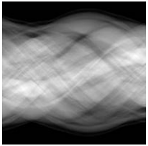
Inverse Problems

Given: Observations (measurements) $\mathbf{y}$
Wanted: Cause for the measured observations $\mathbf{x}$

Inverse problems occur in almost all natural sciences, when the searched for quantity can not be directly observed but has impact on accessible measurements.



Inverse Problems

In the following:

$\mathbf{A}$ denotes the **linear forward model**, which describes how the observations emerge. This means for given observations $\mathbf{y}$, the inverse problem consists in solving the equation

$$\mathbf{A}(\mathbf{x}) = \mathbf{y}.$$

Ill-posed Problems

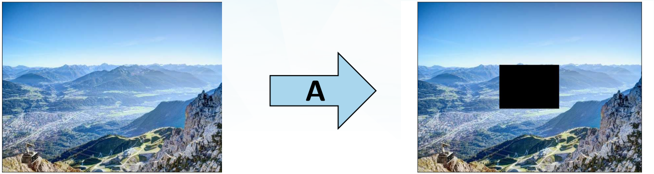
Definition

The problem of solving $\mathbf{A}(\mathbf{x}) = \mathbf{y}$ is called ill-posed, if one of the following is true:

1) There exists no solution of the equation
2) The solution is not unique
3) The solution does not continuously depend on the data $\mathbf{y}$ (Instability).

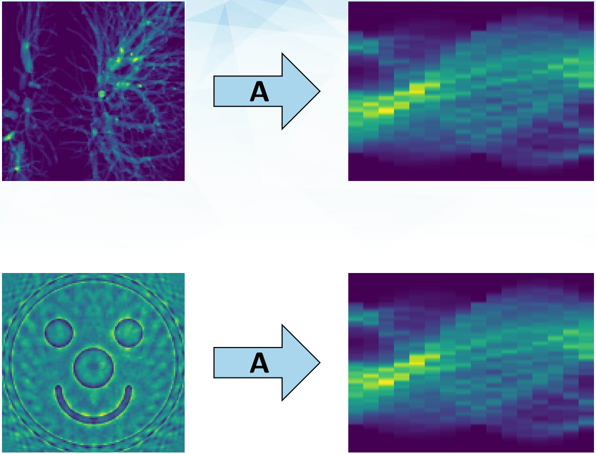
2) Non Uniqueness

A masking operator.



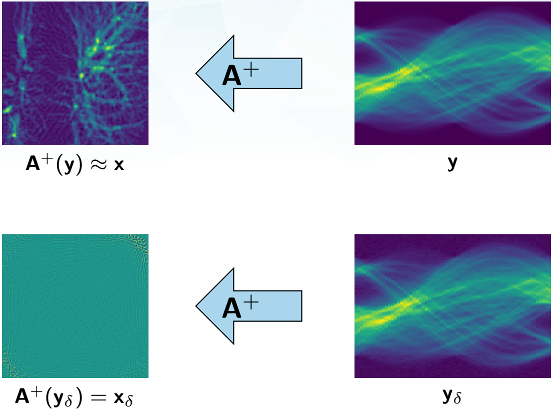
2) Non Uniqueness

A Radon transform (CT).



3) Instability

We consider imperfect data $\mathbf{y}_\delta$, which has distance δ to the perfect data $\mathbf{y}$ ($\|\mathbf{y}_\delta - \mathbf{y}\| = \delta$).
Small perturbations in the data can lead to major deviations in the reconstruction.



Regularization

Regularization methods adress all the issues of ill-posed equations by solving a similar but well-posed problem.

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix},$$

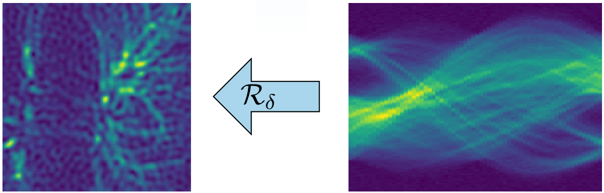
$$\textbf{possible regularized inverse: } \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Regularization

The regularized solution is written as

$$\mathbf{x}_\delta = \mathcal{R}_\delta(\mathbf{y}_\delta).$$

Note that the approximate well-posed equation depends on the level δ of data perturbation.



$\mathcal{R}_\delta(\mathbf{y}_\delta) = \mathbf{x}_\delta$ $\qquad$ $\mathbf{y}_\delta$

Post Processing Networks

Classical post processing networks consist of two steps

- Compute the regularized solution $\mathbf{x}_\delta = \mathcal{R}_\delta(\mathbf{y}_\delta)$
- Apply a convolutional neural network (CNN) Φ to improve the reconstruction $\hat{\mathbf{x}} = \Phi(\mathbf{x}_\delta)$.

Typically $\hat{\mathbf{x}}$ is improved visually but it is **not guaranteed that it still explains the data.**

Null Space Networks

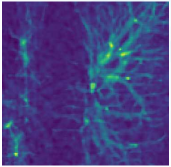
Idea: Allow the CNN Φ_N to only change parts in the solution that do not change the data

$$\hat{\mathbf{x}} = \mathcal{R}_\delta(\mathbf{y}_\delta) + \underbrace{\Phi_N(\mathcal{R}_\delta(\mathbf{y}_\delta))}_{\mathbf{A}(\Phi_N(\mathcal{R}_\delta(\mathbf{y}_\delta))=0} .$$

Schwab, J., Antholzer, S. & Haltmeier, M. "Deep null space learning for inverse problems: convergence analysis and rates." Inverse Problems (2018).

Null Space Networks

$\hat{\mathbf{x}}$



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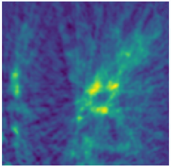
$\mathcal{R}_\delta(\mathbf{y}_\delta)$



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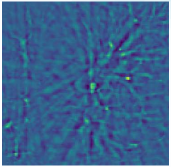
$\Phi_N(\mathcal{R}_\delta(\mathbf{y}_\delta))$



where $\mathbf{A}($  $) = 0.$

Regularizing Networks

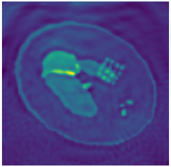
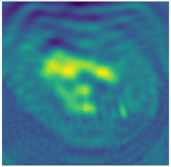
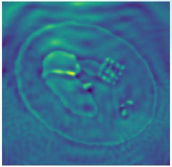
Generalization of deep null space networks. Network depends on the noise level

$$\hat{\mathbf{x}} = \mathcal{R}_\delta(\mathbf{y}_\delta) + \underbrace{\Phi_N^\delta(\tilde{\mathcal{R}}_\delta(\mathbf{y}_\delta))}_{\mathbf{A}\left(\Phi_N^\delta(\tilde{\mathcal{R}}_\delta(\mathbf{y}_\delta)\right) \leq \varepsilon(\delta)} .$$

Schwab, J., Antholzer, S. & Haltmeier, M. "Big in Japan: Regularizing Networks for Solving Inverse Problems." Journal of Mathematical Imaging and Vision (2019)

Regularizing Networks

$\hat{\mathbf{x}}$ = $\mathcal{R}_\delta(\mathbf{y}_\delta)$ + $\Phi_N^\delta(\tilde{\mathcal{R}}_\delta(\mathbf{y}_\delta))$

 $=$  $+$ 

where $\mathbf{A}($  $)$ is small depending on the noise level δ.

Schwab, J., Antholzer, S., Nuster, R., Paltauf, G., & Haltmeier, M. "Deep Learning of truncated singular values for limited view photoacoustic tomography." Photons Plus Ultrasound: Imaging and Sensing 2019

Conclusion and Outlook

- Introduction of data consistent post processing networks
- Convergence analysis of deep learning supported reconstruction methods
- Quantitative error estimates

Current and future work:

- Generalization to non-linear inverse problems
- Implementation for real tomographic devices