Starred exercises are optional.

1) a) Let $f(n) = n + \sqrt{n}$ and $g(n) = n^3 + 2n^2 + 1$. Use $\limsup_{n\to\infty} \frac{f(n)}{g(n)}$ and $\liminf_{n\to\infty} \frac{f(n)}{g(n)}$ to determine each of $f \in O(g), \Omega(g), \Theta(g)$.

   b) Let $f, g, h : \mathbb{N} \to [0, \infty)$. Give a proof or a counterexample for the following statements:
   - If $f, g \in \Omega(h)$ and $g(n) > 0$ for all $n \in \mathbb{N}$, then $\frac{f}{g} \in \Omega(h)$
   - if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$

2) Consider the Haskell function mirror:

```
mirror :: BinTree a -> BinTree a
mirror Empty = Empty
mirror (Node tL x tR) = Node (mirror tR) x (mirror tL)
```

   Can the Master Theorem be used to determine the runtime of `mirror`?

   a) Find a counterexample that shows that it is not possible to apply the Master theorem
   b) Propose additional assumptions that allow the application of the theorem
   c) Write the equation for the problem with the additional proposed assumptions, and determine the complexity using the Master theorem

3) Suppose to have the following recurrence $T(n) = T(\frac{n}{2}) + 2 \cdot T(\frac{n}{4}) + 1$ for inputs n of shape 2^k where $k \geq 2$, and $T(n) = 1$ for inputs $n = 2^0 = 1$ and $n = 2^1 = 2$.

   a) Compute the values of T for inputs $n = 2^k$ where $k \in \{0, 1, \ldots, 10\}$.
   b) Argue that a closed form for T cannot be found by applying the Master Theorem.
   c) i) Either guess a closed form for T (try to see a pattern in the values computed in a)) and verify that it's correct;
      ii) Or prove that for all k, $T(2^k) \leq 2^k$.
      Hint: show that if k is even then $T(2^k) \leq 2^k$ and if k is odd then $T(2^k) \leq 2^k - 1$.

   Clearly indicate your choice among these two items.

4*) Answer one of the following two items, clearly indicating your choice.

   i) Give monotonically increasing functions $f, g : \mathbb{N} \to [0, \infty)$ with $f \notin O(g)$ and $g \notin O(f)$, where a function h is *monotonically increasing* if for all x, y, if $x \leq y$ then $h(x) \leq h(y)$.[1] Are there countably infinitely many such functions? That is, are there monotonically increasing $f_i : \mathbb{N} \to [0, \infty)$ for $i \in \mathbb{N}$ that are all incomparable, i.e. such that if $i \neq j$, then $f_i \notin O(f_j)$? If so, give such functions. If not, argue why.

   ii) Rivest (the R in RSA) constructed the crypto-puzzle LCS35 in 1999, expecting that it would take 35 years to decrypt. It was decrypted last year (twice, independently, by a person first and one day later by a group of people). Give the exact problem statement, explain what makes/made the problem difficult, and how knowing the factors of the product of primes in it would make the problem easier.

[1] Note that the functions $x \mapsto \sin(x) + 1$ and $x \mapsto \cos(x) + 1$ as discussed in the lecture, only satisfy 'half' the specification as they are not monotonically increasing.