Discrete Structures/Mathematics WS 2020/2021 LVA 703069

EXAM 2 February 24th, 2021

This exam consists of <u>three</u> regular exercises (1–3) each worth 20 points, so 60 points in total. In addition, there are bonus exercises (1(d),2(d),3(d) worth 14 points in total). The available points for each item are written in the margin. You need at least 30 points to pass. Always explain your answer. In particular, for yes/no questions the correct answer is worth 1 point with the remaining points for the explanation. The time available is 1 hour and 45 minutes (105 minutes).

Throughout this exam, let m be your Matrikelnr. having 8 digits $m_1m_2m_3m_4m_5m_6m_7m_8$ with $0 \leq m_i \leq 9$.

Start each document handed in with (writing down) your name and m.

$\boxed{1}$ (a) Let G be a directed graph with nodes $\{a, b, c, d\}$ and labeled edges

$$\{(a, m_2, b), (a, m_3, c), (a, m_4, d), (b, m_5, a), (b, m_6, a), (c, m_7, d), (d, m_8, c)\}$$

[8] where each triple describes the start of the edge, the weight, and the end of the edge. Use Floyd's algorithm to compute the distances (least weight among possible paths) between all nodes. Give the start matrix and all intermediate matrices, and give the distance from b to d.

(b) Let G be the graph with nodes $\{a, b, c, d, e, f, g\}$ and edges

$$\{a, b\}, \{a, e\}, \{b, e\}, \{c, d\}, \{c, g\}, \{d, g\}, \{d, f\}, \{f, g\}$$

[8] The weights of the edges in this order are $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8$. Use Kruskal's algorithm to compute a spanning forest with minimal weight. Show all the intermediate steps and the final spanning forest.

(c) Consider the relation R on digits $\{0, \ldots, 9\}$, defined by $R(d, e)$ if and only if the digit d does not appear later than the digit e in the word $m_1m_2m_3m_4m_5m_6m_7m_8$. For example, in the word 121, the digit 2 does

[4] not appear later than 2 and 3 does not appear later than 4, but 1 does appear later than 1, 1 appears later than 2, and 2 appears later than 1.

Is R an equivalence relation? If so, prove it. If not, give for **each** property (among the 3 properties equivalence relations have) that is not satisfied a counterexample.

[4] (d) (bonus) Propose a non-empty well-founded strict order on finite trees, and show that it is indeed a well-founded strict order.

$\boxed{2}$ (a) Let $f(n) = m_8 \cdot n^2 + m_7 \cdot n + m_6 + 1$.

- [6] Determine $s \geq 0$ such that $f(n) \in \Theta(n^s)$ and show that this relation holds. Base your answer on the definition of Θ, give intermediate steps and mention results (from the lecture) used. However, simple convergence results (as for $\frac{1}{n^i}$ with i a positive number) can be used without proof.
- [6] Let $T : \mathbb{N} \to \mathbb{N}$ be an increasing function satisfying $T(n) = (m_8 + 1) \cdot T(\frac{n}{2}) + f(n)$ for $n = 2^k$ with k a positive natural number and $T(1) = 1$. Determine a closed-form expression e, such that e is an upper and lower bound for T.

(b) Let k be the number $m_1 m_2$ and k' be $m_2 m_1$ in decimal notation.

[4] Let $V = \{v_1, \ldots, v_k, w_1, \ldots, w_{k'}\}$. Find an undirected graph $G = (V, E)$ with $E \subseteq \{\, (v_i, w_j) \mid 1 \leq i \leq k, 1 \leq j \leq k' \,\}$ such that $\sum_{i=1}^{k} \deg(v_i) = 2 + \sum_{j=1}^{k'} \deg(w_j)$ or prove that such a graph cannot exist.

[4] (c) Prove the following statement if it is true or give a counterexample if it is false: Let M, N be countable such that $M \subseteq N$, then $N \setminus M$ is finite.

[5] (d) (bonus) Let L_1, L_2 be languages such that L_1 reduces to L_2 by a computable function f, i.e. $L_1 \leq L_2$. Show that L_1 is recursive if L_2 is recursive, by providing a total Turing Machine accepting L_1 using f.

$\boxed{3}$ (a) Let k be the number $m_3 m_4$ and k' be $m_5 m_6$ in decimal notation.

[5] Compute $lcm(k, k')$ by first computing $\gcd(k, k')$ by means of the Euclidean algorithm, giving the intermediate steps. (You may choose either the subtraction-based or the division-based version; indicate which version you use.)

(b) Let k be the number $2 + m_2$ in decimal notation.

[10] Consider all functions from the set $N = \{n \mid 0 \leq n < k \cdot (k+1)\}$ of natural numbers to the set $P = \{(x, y) \mid 0 \leq x < k, 0 \leq y < k + 1\}$ of pairs of natural numbers.

- Show that $\#N = \#P$ by giving some bijection between N and P.
- How many injective functions from N to P are there? Since the number is typically large, it suffices to explain how it is computed.
- Is the function $f(n) = (n \mod k, n \mod (k + 1))$ a bijection from N to P? If so, explain why. If not, show how bijectivity fails.

[5] (c) Let k be your matrikelnr. $m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8$ in decimal notation. Does there then exist an inverse of k modulo 15? If so, compute the inverse. If not, give a proof. (You may use a computer/calculator for intermediate steps, say for modulo computations, but you have to explain your method.)

[5] (d) (bonus) Assume k is a natural number greater than 2, and let n be a number written in base-k as $n_\ell \ldots n_0$ with $0 \leq n_i < k$. Prove $n \equiv \sum_{i=0}^{\ell} n_i \pmod{k-1}$.

This provides a method to easily check whether n is divisible by $k-1$ by checking whether the sum $\sum_{i=0}^{\ell} n_i$ is divisible by $k-1$ (it may be repeated for intermediate sums). Two examples of the method: We check the base-10 (decimal) number 1872 is divisible by 9 by the chain $1+8+7+2 \equiv 9+7+2 \equiv 16+2 \equiv 1+6+2 \equiv 0 \pmod 9$. The base-3 number 122 is odd (not divisible by 2) as checked by $1+2+2 \equiv 1 \pmod 2$; indeed $(122)_3$ is 17 in decimal notation so odd.