universität innsbruck

Discrete Structures/Mathematics WS 2020/2021 LVA 703069

EXAM 3 September 24th, 2021

This exam consists of <u>three</u> regular exercises (1–3) each worth 20 points, so 60 points in total. In addition, there are bonus exercises (1(e) and 2(d) worth 11 points in total). The available points for each item are written in the margin. You need at least 30 points to pass. Always explain your answer. In particular, for yes/no questions the correct answer is worth 1 point with the remaining points for the explanation. The time available is 1 hour and 45 minutes (105 minutes).

Throughout, let m be your Matrikelnr. having 8 digits $m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8$ with $0 \leq m_i \leq 9$. **Start each document handed in with (writing down) your name and m.**

1 Consider the relation R on the set $D = \{1, 4, 5, 7, 9\}$ of digits, given by $i \; R \; j$ if i occurs strictly before j in your Matrikelnr.. For example, for the Matrikelnr. 15658571 we would obtain: $1 \; R \; 1$, $1 \; R \; 5$, $1 \; R \; 7$, $5 \; R \; 1$, $5 \; R \; 5$, $5 \; R \; 7$ and $7 \; R \; 1$, with no other pair of elements of D being R-related. (Note that the questions below are about R for *your* Matrikelnr.).

[3] (a) Is R irreflexive? Explain your answer.

[3] (b) Is R well-founded? Explain your answer.

[7] (c) Compute the reflexive–transitive closure R^* of R. Illustrate the algorithm you use by giving (some of) the intermediate computation steps.

[7] (d) Draw the directed graph G corresponding to R and compute for each pair of nodes in G the length of the shortest path (you may assume edges to have weight 1) from the first to the second. Illustrate the algorithm you use by giving (some of the) intermediate computation steps.

[5] (e) (bonus) Does there exist a Martikelnr. for which the corresponding relation R would *not* be transitive? Argue why (not).

2 (a) For $k = m_7 + 2$, consider k-mergesort, the variation on mergesort that sorts a list ℓ of length greater than 1 by splitting it into k lists $\ell_1, \ldots, \ell_k$ of equal lengths, recursively k-mergesorts $\ell_1, \ldots, \ell_k$ to yield $s_1, \ldots, s_k$, and then does a k-way merge of the $s_1, \ldots, s_k$ to yield a sorted list s.

[7] Analyse the (time) complexity $T(n)$ of k-mergesort, for n the length of the input-list. You may restrict your analysis to lists whose length is a power of k (so that in each recursive call all k parts indeed do have equal

lengths), and you may assume that a k-way merge takes time linear in the length of the merged lists.

[7] (b) Does there exist a natural number x such that $x \equiv m_2 \pmod{m_3+2}$ and $x \equiv m_4 \pmod{m_3+3}$? If so, compute such an x by applying Bézout's Lemma/the Chinese Remainder Theorem. If not, argue why not.

[6] (c) Do there exist sets A, B and C such that $\#(A \cup B \cup C) = 20$, $\#A = 10$, $\#B = 10$, $\#C = 10$, $\#(A \cap B) = m_5$, $\#(B \cap C) = m_6$, $\#(C \cap A) = m_7$ and $\#(A \cap B \cap C) = m_8$? Prove your answer.

[6] (d) (bonus) Recall that for languages L_1, L_2 over Σ, L_1 is said to be *reducible* to L_2, denoted by $L_1 \leq L_2$, if there exists a computable $f : \Sigma^* \to \Sigma^*$ such that $x \in L_1 \Leftrightarrow f(x) \in L_2$.

Explain how the notion of reducibility is typically used to show languages are not recursive, and why the condition that f be *computable* cannot be omitted from the definition (of reducibility), without rendering it useless for that usage.

3 Determine for each of the following statements whether they are true or not. (Recall from above that a correct yes/no answer gives 1 point; the remaining 3 points are for the explanation.)

[4] (a) Let $k = m_8 + 2$ (with m_8 taken from your Matrikelnr.) and suppose p is a natural number relatively prime to k such that $k^{p-1} \not\equiv 1 \pmod{p}$. Then p is not a prime number.

[4] (b) Every undirected connected multigraph having exactly 4 edges e_3, e_4, e_5, e_6 with respective weights m_3, m_4, m_5, m_6 (where the m_i are taken from your Matrikelnr.), has a *unique* minimum spanning tree.

[4] (c) For all finite sets A, B, C and D with respective cardinalities m_4, m_5+1, m_6 and m_7 (where the m_i are taken from your Matrikelnr.), the cardinality of the set $A^B \times (C \cup D)$ can be computed as $m_4^{m_5+1} \cdot (m_6 + m_7)$.

[4] (d) If removing an edge from an undirected (and unweighted) graph G results in G', then we have for their diameters $d(G) \leq d(G')$.

Here the *diameter* $d(G)$ of a graph G is the maximum of the distances of all pairs of vertices in G, where for vertices v and w their *distance* $d(v, w)$ is defined as usual as the length of the shortest path in G between v and w (∞ if no such path exists; thus, $d(G) = \infty$ iff G is not connected).

[4] (e) Let k be the number $m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 8$ in decimal notation, i.e. your Matrikelnr. followed by an 8. Then k is not a square.