Starred exercises are optional.

1) a) Compute the greatest common divisor of 42 and 55, first with the difference based Euclidean Algorithm and second with the remainder based variant of the algorithm. In how many steps does each version reach the final result?

b) We call an integer a *relatively* prime to an integer b, if their greatest common divisor is 1. Show that any natural number $a > 0$ is relatively prime to its successor $a + 1$ and show in how many steps each variant of the Euclidean Algorithm would terminate. Is the relation $R = \{(a, b) \mid a \text{ relatively prime to } b\}$ over $\mathbb{Z}$ reflexive, symmetric, transitive?

c) Prove that $\gcd(\gcd(a, b), c) = \gcd(a, \gcd(b, c))$ holds for all $a, b, c \in \mathbb{Z}$. Hint: show that either side divides the other by showing it divides each of a,b and c (repeatedly using the definition of gcd), and the greatest such divisor is unique (if $d \mid d'$ and $d' \mid d$ then $d = d'$).

2) a) Let $A = \{x \in \mathbb{R} \mid r \geq 0\}$ be the set of non-negative reals numbers. Visualise in the XY-plane (i.e. using Cartesian coordinates) the equivalence class (x, x) is in, for each $x \in \{1, 2, 5\}$ and each of the following equivalence relations R_i on $A \times A$:

- $(x, y)\ R_1\ (x', y')$ if $x + y' = x' + y$;
- $(x, y)\ R_2\ (x', y')$ if $x^2 + y^2 = x'^2 + y'^2$;
- $(x, y)\ R_3\ (x', y')$ if $x + y = x' + y'$; and
- $R_4 := R_2 \cap R_3$.

b) Consider the alphabet $\Sigma = \{a, \dots, z\}$ ordered as usual by $<$, i.e. $a < \dots < z$. Define the relation R on words in Σ^* by $v\ R\ w$ if for every letter x in v there is a letter y in w such that $x \leq y$. Then R is a quasi-order (it is reflexive and transitive) so induces an equivalence relation $R \cap R^{-1}$ (see the slides). How many equivalence classes are there?

3) List the following six sets in order of increasing (non-decreasing) cardinality. (1) the set S_1 of prime numbers; (2) the set S_2 of all singleton sets $\{x\}$ with $x \in \mathbb{R}$; (3) the set S_3 of all Java programs that print the text of the Austrian constitution; (4) the set S_4 of all real numbers that are not rational, i.e. $\mathbb{R} - \mathbb{Q}$; (5) the set S_5 of all subsets of people on earth (living at the moment you make the exercise). (6) the set S_6 of all finite graphs having as nodes strings over the alphabet $\{a, b\}$ and as edges pairs of nodes.

Explain each of your orderings (or in case of sets having the same cardinality, both directions), and determine for each set whether it is finite and/or countable or not.

4*) Consider the function $f : \mathbb{N} \to \mathbb{B}^*$ that given a natural number returns the word encoding of that number in unary, that is $f(n) = 1^n$. Consider also the function $g : \mathbb{B}^* \to \mathbb{N}$ that given a word w returns the natural number encoded in binary[1] prefixed by 11, that is $(11w)_2$. So for example $g(\epsilon) = (11)_2 = 3$ and $g(00) = (1100)_2 = 12$.

Both these functions are injections but not bijections. The proof of the Schröder–Bernstein theorem given in the slides of the 8th lecture gives a way to build a bijection using these two functions. Describe this bijection in words (or in Haskell).

[1] In this exercise we use $(1101)_2$ to denote the value of the binary number 1101, i.e. 13.