Homework

Mark your solved exercise in OLAT and also upload your solution as a PDF before January 13, 6am.

1. Consider the following algorithm for computing the digit sum of a number w.r.t. base 8.

   ```
   void octalSum(int n) {
     int ds = 0;
     while (n > 0) {
        ds := ds + (n mod 8);
        n  := n div 8;           // = floor (n / 8)
     }
     return sum;
   }
   ```

   Write down a LIA formula φ that approximates one loop iteration, i.e., where division and modulo are approximated. Figure out a suitable expression $e(n, ds)$ and a constant c. Write down the two LIA constraints that have to be unsatisfiable in order to show termination. Are these constraints satisfiable over $\mathbb{Z}$ or over $\mathbb{Q}$? (Yes / No suffices) (2 pt.)

2. Consider the linear inequalities:

   $$x_1 - 5x_2 > 6 \qquad\qquad 2x_1 + 3x_2 \leqslant 2$$

   (a) Convert the inequalities into matrix form $A\vec{x} \leqslant \vec{b}$ and determine a bound on the numbers of an integral solution. (1 pt.)

   (b) Apply the branch-and-bound algorithm to find an integral solution, or detect unsatisfiability. You can use the non-incremental simplex implementation from the previous exercise sheet. (1 pt.)

(c) Represent $\{\vec{x} \mid A\vec{x} \leqslant \vec{b}\}$ as $hull(X) + cone(V)$ by following the approach in slides 22 – 25. (2 pt.)

Hint: $\begin{pmatrix} -1 \\ 5 \\ 7 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -31 \\ 12 \\ -13 \end{pmatrix}$

(d) In the previous part you will have computed

$$\{\vec{x} \mid A\vec{x} \leqslant \vec{b}\} = cone\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -5 \\ -1 \end{pmatrix}\right\} + hull\left\{\begin{pmatrix} \frac{31}{13} \\ -\frac{12}{13} \end{pmatrix}\right\}.$$

Use this equality to derive an improved bound on an integral solution, in particular one that is better than the one in part (a). Can you derive different bounds for x_1 and x_2? (1 pt.)

3. Consider the construction to prove the decomposition theorem on slide 22.

(a) On the slide it is just assumed that all $\tau_i > 0$. Prove that this assumption can be made by showing that the last component of any vector in C is always non-negative. (1 pt.)

(b) Prove $P \subseteq cone(X) + hull(V)$. (2 pt.)

For both tasks you may assume the Farkas–Minkowsky–Weyl theorem and don't have to prove it.