universität innsbruck

Constraint Solving	WS 2021	LVA 703305
Week 12		January 20, 2022

Homework

Mark your solved exercise in OLAT and also upload your solution as a PDF before January 20, 6am.

1. We consider the complexity of quantifier free linear integer arithmetic.
   (a) In the lecture it was shown that satisfiability of conjunctions of linear inequalities over $\mathbb{Z}$ is in NP. Argue that satisfiability of arbitrary Boolean combinations of linear inequalities over $\mathbb{Z}$ is still in NP. (1 pt.)
   (b) Show that satisfiability of conjunctions of linear inequalities over $\mathbb{Z}$ is NP-hard. (2 pt.)
   Hint: Use a reduction from the NP-hard problem to determine whether a given conjunctive normal form over propositional variables is satisfiable.

2. Consider a set of linear inequalities formulated as $A\vec{x} \leqslant \vec{b}$ with $A \in \mathbb{Z}^{n \times m}$ and $b \in \mathbb{Z}^n$. The small model property states that if there is an integer solution, then there also is an integer solution $\vec{x}$ such that all entries of $|\vec{x}|$ are at most $(n+1)! \cdot c^n$, assuming that all entries e in A and $\vec{b}$ satisfy $|e| \leqslant c$ for some positive integer c.
   Does the small model property still hold, if one weakens the assumptions
   (a) from $A \in \mathbb{Z}^{n \times m}$ and $\vec{b} \in \mathbb{Z}^n$ to $A \in \mathbb{Q}^{n \times m}$ and $\vec{b} \in \mathbb{Z}^n$? (1 pt.)
   (b) from $A \in \mathbb{Z}^{n \times m}$ and $\vec{b} \in \mathbb{Z}^n$ to $A \in \mathbb{Z}^{n \times m}$ and $\vec{b} \in \mathbb{Q}^n$? (2 pt.)
   In both cases, argue why this is the case or provide a counter-example.

3. Compute a Gomory cut given the following result of the simplex algorithm.
   - tableau equation $x_4 = x_1 - 2\frac{1}{2}x_2 + 2x_3$
   - $v(x_1) = 1$, $v(x_2) = -\frac{1}{2}$, $v(x_3) = \frac{1}{2}$, and $v(x_4) = 3\frac{1}{4}$
   - x_1 is at its lower bound, x_2 and x_3 are at their upper bound

   In particular, write down the constant c, the sets L^+, L^-, U^+, U^-, as well as the final inequality. (1 pt.)

4. Extend the algorithm for difference logic to arbitrary constraints, i.e., including strict inequalities, such as the following ones.

$$x - y < -3 \qquad x - w \leqslant -4 \qquad y - z < 5 \qquad z - w \leqslant -2 \qquad z - x \leqslant -1$$

(a) Show how strict inequalities can be added for $\mathbb{Z}$ and test it on the example constraints. (1 pt.)

(b) Show how strict inequalities can be added for $\mathbb{Q}$ and test it on the example constraints. (2 pt.)