WS 2021 lecture 11

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

Properties of DPLL(T) Simplex Algorithm

- termination ensured via Bland's rule:
  choose x_i and x_j for pivoting in a way that $(x_i, x_j) \in B \times N$ is lexicographically smallest
- worst-case complexity is exponential, but only on artificial examples
- provides incremental interface (activation flags for bounds) and unsatisfiable cores
  (Haskell: initSimplex, assert *i*, check, solution, checkpoint, backtrack *cp*)
- strict inequalities supported, but requires arithmetic using $\mathbb{Q}_\delta$

$$x < c \implies x \leq c - \delta$$
$$x > c \implies x \geq c + \delta$$

- decides quantifier-free conjunctions for LRA
- not well suited for linear programming, i.e., optimization problems

Outline

Example (Application of Linear Arithmetic: Termination Proving)

- consider program (assuming that `int` behaves like mathematical integers)

```
int factorial(int n) {
  int i = 1;
  int r = 1;
  while (i < n) {
    i = i + 1;
    r = r * i;  }
  return r;          }
```

- φ describes one iteration of loop (primed variables store values after iteration)

$$\varphi := i < n \land i' = i + 1 \land r' = r \cdot (i + 1) \land n' = n$$

- proving termination: find expression $e(i, n, r)$ and integer c such that
  - $\varphi \longrightarrow e(i, n, r) \geq e(i', n', r') + 1$ (expression decreases in every iteration)
  - $\varphi \longrightarrow e(i', n', r') \geq c$ (expression is bounded from below by c)

Example (Termination Proof Continued)

- loop iteration $\varphi := i < n \land i' = i + 1 \land r' = r \cdot (i + 1) \land n' = n$
- proving termination by validity of formulas

$$\varphi \longrightarrow e(i, n, r) \geq e(i', n', r') + 1 \qquad \varphi \longrightarrow e(i', n', r') \geq c$$

- is equivalent to unsatisfiability of negated formulas

$$\varphi \land e(i, n, r) < e(i', n', r') + 1 \qquad \varphi \land e(i', n', r') < c$$

- choosing $e(i, n, r) := n - i$ and $c := -1$, and dropping all non-linear constraints yields two LIA problems:
  - $i < n \land i' = i + 1 \land n' = n \land n - i < n' - i' + 1$ ($\neg$ decrease)
  - $i < n \land i' = i + 1 \land n' = n \land n' - i' < -1$ ($\neg$ bounded)

  both problems are unsatisfiable over $\mathbb{R}$ (just run simplex), so termination is proved

Example (Application of Linear Integer Arithmetic: Termination Proving)

- consider another program

```
int log2(int x)      {
    int n := 0;
    while (x > 0) {
      x := x div 2;
      n := n + 1; }
    return n;        }
```

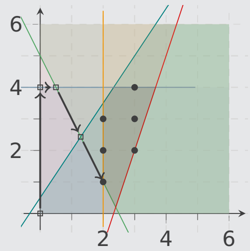
- $\varphi := x > 0 \land 2x' \leq x \land x \leq 2x' + 1 \land n' = n + 1$
- choose $e(x, n) = x$ and $c = -1$; obtain two LIA problems that should be unsatisfiable
  - $\varphi \land x < x' + 1$ ($\neg$ decrease)
  - $\varphi \land x' < -1$ ($\neg$ bounded)
- ($\neg$ bounded) is unsatisfiable over $\mathbb{R}$
- ($\neg$ decrease) is unsatisfiable over $\mathbb{Z}$, but not over $\mathbb{R}$ $\Longrightarrow$ require LIA solver
- remark: LIA reasoning is crucial, the problem is not wrong choice of expression e; program does not terminate when executed with real number arithmetic

Outline

Example

$3x - 2y \geq -1$
$y \leq 4$
$2x + y \geq 5$
$3x - y \leq 7$

- looking for solution in $\mathbb{Z}^2$
- infinite $\mathbb{R}^2$ solution space, six solutions in $\mathbb{Z}^2$
- simplex returns $(\frac{9}{7}, \frac{17}{7})$



Branch and Bound, a Solver for LIA Formulas – Idea

- add constraints that exclude current solution in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ but do not change solutions in $\mathbb{Z}^2$
- in current solution $1 < x < 2$, so use simplex on two augmented problems:
  - $C \wedge x \leqslant 1$ unsatisfiable
  - $C \wedge x \geqslant 2$ satisfiable, simplex can return $(2, 1)$

Algorithm BranchAndBound(φ)

Input: LIA formula φ, a conjunction of linear inequalities
Output: unsatisfiable, or satisfying assignment

let *res* be result of deciding φ over $\mathbb{R}$ ▷ e.g. by simplex
if *res* is **unsatisfiable then**
  return **unsatisfiable**
else if *res* is solution over $\mathbb{Z}$ **then**
  return *res*
else
  let x be variable assigned non-integer value q in *res*
  $res = \text{BranchAndBound}(\varphi \wedge x \leq \lfloor q \rfloor)$
  if $res \neq$ **unsatisfiable then**
    return *res*
  else
    return $\text{BranchAndBound}(\varphi \wedge x \geq \lceil q \rceil)$

Example (Termination Proof of log2, Continued)

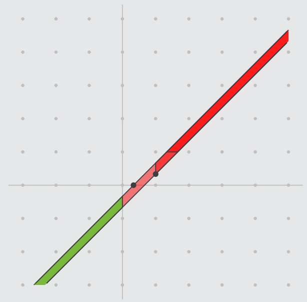
- problematic formula (satisfiable over $\mathbb{R}$)

$$\psi := x > 0 \wedge 2x' \leq x \wedge x \leq 2x' + 1 \wedge x < x' + 1 \qquad (\neg\text{ decrease})$$

- execution of BranchAndBound on ψ (short notation: $BB(\psi)$)
  - simplex: $v(x) = 1$, $v(x') = \frac{1}{2}$
  - invoke $BB(\psi \wedge x' \geq 1)$, simplex: unsatisfiable
  - invoke $BB(\psi \wedge x' \leq 0)$, simplex: $v(x) = \frac{1}{2}, v(x') = -\frac{1}{4}$
    - invoke $BB(\psi \wedge x' \leq 0 \wedge x \geq 1)$, simplex: unsatisfiable
    - invoke $BB(\psi \wedge x' \leq 0 \wedge x \leq 0)$, simplex: unsatisfiable
  - return unsatisfiable

Example (Branch and Bound – Problem)

consider $\psi := 1 \leq 3x - 3y \wedge 3x - 3y \leq 2$



- $v(x) = \frac{1}{3}$, $v(y) = 0$, add $x \leq 0$ or $x \geq 1$
- for $\psi \wedge x \geq 1$: $v(x) = 1$, $v(y) = \frac{1}{3}$, add $y \leq 0$ or $y \geq 1$
- ... *BranchAndBound* is not terminating, since search space is unbounded

Theorem (Small Model Property of LIA)

if LIA formula ψ has solution over $\mathbb{Z}$ then it has a solution v with

$$|v(x)| \leq \mathit{bound}(\psi) := (n+1)! \cdot c^n$$

for all x where

- *n: number of variables in ψ*
- *c: maximal absolute value of numbers occurring in ψ*

Consequences and Remarks

- satisfiability of ψ for LIA formula is in NP
- invoke

$$\mathit{BranchAndBound}\left(\psi \wedge \bigwedge_{x \in \mathit{vars}(\psi)} -\mathit{bound}(\psi) \leq x \leq \mathit{bound}(\psi)\right)$$

  to decide solvability of ψ over $\mathbb{Z}$
- bound is quite tight: $c \leq x_1 \wedge c \cdot x_1 \leq x_2 \wedge \ldots \wedge c \cdot x_{n-1} \leq x_n$ implies $x_n \geq c^n$

Outline

Geometric Objects

- polytope: convex hull of finite set of points X

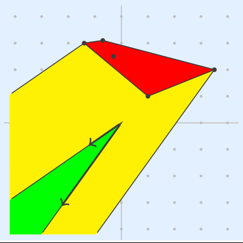
$$\mathit{hull}(X) = \{\lambda_1\vec{v}_1 + \ldots + \lambda_m\vec{v}_m \mid \{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq X \wedge \lambda_1, \ldots, \lambda_m \geq 0 \wedge \sum \lambda_i = 1\}$$

- finitely generated cone: non-negative linear combinations of finite set of vectors V

$$\mathit{cone}(V) = \{\lambda_1\vec{v}_1 + \ldots + \lambda_m\vec{v}_m \mid \{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq V \wedge \lambda_1, \ldots, \lambda_m \geq 0\}$$

- polyhedron: polytope + finitely generated cone

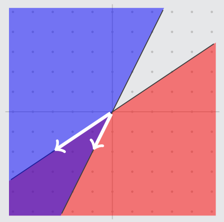
$$\mathit{hull}(X) + \mathit{cone}(V) = \{\vec{x} + \vec{v} \mid \vec{x} \in \mathit{hull}(X) \wedge \vec{v} \in \mathit{cone}(V)\}$$



More Geometric Objects

- C is polyhedral cone iff $C = \{\vec{x} \mid A\vec{x} \leq \vec{0}\}$ for some matrix A
  iff C is intersection of finitely many half-spaces

Example



Theorem (Farkas, Minkowski, Weyl)

A cone is polyhedral iff it is finitely generated.

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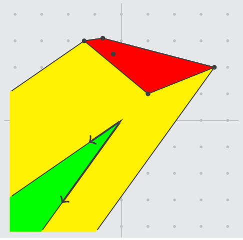
Theorem (Decomposition Theorem for Polyhedra)

A set $P \subseteq \mathbb{R}^n$ can be described as a polyhedron $P = hull(X) + cone(V)$ for finite X and V iff $P = \{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ for some matrix A and vector $\vec{b}$.
Moreover, given X and V one can compute A and $\vec{b}$, and vice versa.

Example



Proof Idea of Small Model Property

1. convert conjunctive LIA formula ψ into form $A\vec{x} \leq \vec{b}$
2. represent polyhedron $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ as polyhedron $P = hull(X) + cone(V)$
3. show that P has small integral solutions, depending on X and V
4. approximate size of entries of vectors in X and V to obtain small model property

Remark

- given ψ, one can compute X and V instead of using approximations
- however, this would be expensive: decomposition theorem requires exponentially many steps (in n, m) for input $A \in \mathbb{Z}^{m \times n}$ and $\vec{b} \in \mathbb{Z}^m$

Step 1: Conjunctive LIA Formula into Matrix Form $A\vec{x} \leq \vec{b}$

- (variable renamed) formula

$$x_1 > 0 \qquad 2x_2 \leq x_1 \qquad x_1 \leq 2x_2 + 1 \qquad x_1 < x_2 + 1$$

- eliminate strict inequalities (only valid in LIA)

$$x_1 \geq 0 + 1 \qquad 2x_2 \leq x_1 \qquad x_1 \leq 2x_2 + 1 \qquad x_1 + 1 \leq x_2 + 1$$

- normalize (only $\leq$, constant to the right-hand-side)

$$-x_1 \leq -1 \qquad -x_1 + 2x_2 \leq 0 \qquad x_1 - 2x_2 \leq 1 \qquad x_1 - x_2 \leq 0$$

- matrix form

$$\begin{pmatrix} -1 & 0 \\ -1 & 2 \\ 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Step 3: Small Integral Solutions of Polyhedrons

- consider finite sets $X \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{Z}^n$
- define

$$B = \{\lambda_1 \vec{v_1} + \ldots + \lambda_n \vec{v_n} \mid \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V \wedge 1 \geq \lambda_1, \ldots, \lambda_n \geq 0\} \subseteq cone(V)$$

Theorem

$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \emptyset \longleftrightarrow (hull(X) + B) \cap \mathbb{Z}^n = \emptyset$

Corollary

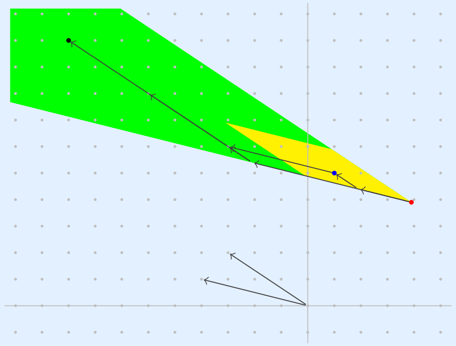
Assume $|c| \leq b \in \mathbb{Z}$ for all entries c of all vectors in $X \cup V$.
Define $Bnd := b \cdot (1 + n)$. Then

$$\begin{aligned} & (hull(X) + cone(V)) \cap \mathbb{Z}^n = \emptyset \\ \longleftrightarrow\ & (hull(X) + cone(V)) \cap \{-Bnd, \ldots, Bnd\}^n = \emptyset \end{aligned}$$

Theorem

$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \emptyset \longleftrightarrow (hull(X) + B) \cap \mathbb{Z}^n = \emptyset$

Proof



Step 2a: Decomposing Polyhedron $P = \{\vec{u} \mid A\vec{u} \leq \vec{b}\}$ into $hull(X) + cone(V)$

1. use FMW to convert polyhedral cone of $\left\{\vec{v} \;\middle|\; \begin{pmatrix} A & -\vec{b} \\ \vec{0} & -1 \end{pmatrix} \vec{v} \leq \vec{0}\right\}$ into $cone(C)$ for integral vectors $C = \left\{\begin{pmatrix}\vec{y}_1 \\ \tau_1\end{pmatrix}, \dots, \begin{pmatrix}\vec{y}_\ell \\ \tau_\ell\end{pmatrix}, \begin{pmatrix}\vec{z}_1 \\ 0\end{pmatrix}, \dots, \begin{pmatrix}\vec{z}_k \\ 0\end{pmatrix}\right\}$ with $\tau_i > 0$ for all $1 \leq i \leq \ell$
2. define $\vec{x}_i := \frac{1}{\tau_i}\vec{y}_i$
3. return $X := \{\vec{x}_1, \dots, \vec{x}_\ell\}$ and $V := \{\vec{z}_1, \dots, \vec{z}_k\}$

Theorem

$P = hull(X) + cone(V)$

Bounds

- the absolute values of the numbers in $X \cup V$ are all bounded by the absolute values of the numbers in C
- hence, bounds on C can be reused to bound vectors in $X \cup V$

Step 2b: Theorem of Farkas, Minkowski, Weyl

A cone is polyhedral iff it is finitely generated.

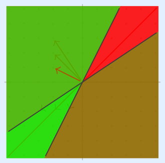
First direction: finitely generated implies polyhedral

- consider $cone(V)$ for $V = \{\vec{v}_1, \dots, \vec{v}_m\} \subseteq \mathbb{R}^n$
- consider every set $W \subseteq V$ of linearly independent vectors with $|W| = n - 1$
- obtain integral normal vector $\vec{c}$ of hyper-space spanned by W
- next check whether V is contained in hyper-space $\{\vec{v} \mid \vec{v} \cdot \vec{c} \leq 0\}$ or $\{\vec{v} \mid \vec{v} \cdot (-\vec{c}) \leq 0\}$
  - if $\vec{v}_i \cdot \vec{c} \leq 0$ for all i, then add $\vec{c}$ as row to A
  - if $\vec{v}_i \cdot \vec{c} \geq 0$ for all i, then add $-\vec{c}$ as row to A
- $cone(V) = \{\vec{x} \mid A\vec{x} \leq \vec{0}\}$
- bounds
  - each normal vector $\vec{c}$ can be computed via determinants

$\Longrightarrow$ obtain bound on numbers in $\vec{c}$ by using known bounds on determinants, cf. slide 25

Example: Construction of Polyhedral Cone from Finitely Generated Cone

$$V = \left\{\begin{pmatrix}-3 \\ -2\end{pmatrix}, \begin{pmatrix}-2 \\ -2\end{pmatrix}, \begin{pmatrix}-1 \\ -2\end{pmatrix}\right\}$$



$$A = \begin{pmatrix} -2 & 3 \\ 2 & -1 \end{pmatrix}$$

- pick $W = \{\vec{w}\}$, $\vec{w} = \begin{pmatrix}-3 \\ -2\end{pmatrix}$ and consider $span\, W$
- compute normal vector $\vec{c} = \begin{pmatrix}-2 & 3\end{pmatrix}$
- if V is in same half-space, add $\pm\vec{c}$ to A

Step 2b: Theorem of Farkas, Minkowski, Weyl

A cone is polyhedral iff it is finitely generated.

Second direction: polyhedral implies finitely generated

- consider $\{\vec{x} \mid A\vec{x} \leq \vec{0}\}$
- define W as the set of row vectors of A
- by first direction obtain integral matrix B such that $cone\,(W) = \{\vec{x} \mid B\vec{x} \leq \vec{0}\}$
- define V as the set of row vectors of B
- $\{\vec{x} \mid A\vec{x} \leq \vec{0}\} = cone\,(V)$
- bounds carry over from first direction

Step 4: Theorem of Farkas, Minkowski, Weyl (bounded version)

Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone, given via an integral matrix A. Let b be a bound for all matrix entries, $b \geq |A_{ij}|$. Then C is generated by a finite set of integral vectors V whose entries are at most $\pm\,(n-1)! \cdot b^{n-1}$.

Outline

Kröning and Strichmann

- Section 5.3

Further Reading

Alexander Schrijver
Theory of linear and integer programming, Chapters 7, 16, 17, and 24
Wiley, 1998.

Important Concepts

- branch-and-bound
- cone (finitely generated or polyhedral)
- decomposition theorem for polyhedra
- Farkas–Minkowski–Weyl theorem
- polyhedron
- small model property of LIA
- termination of program via two validity proofs: decrease and boundedness