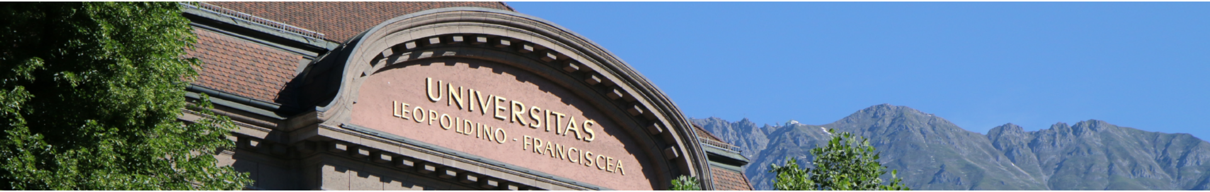
universität innsbruck

WS 2021 lecture 13



Constraint Solving

Cezary Kaliszyk René Thiemann

based on a previous course by Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **Checking Array Bounds**
3. **Array Logic**
4. **Array Properties**
5. **Summary and Further Reading**

Gomory Cuts: Assumptions

- DPLL(T) simplex returned solution v to

$$A\vec{x}_N = \vec{x}_B \qquad (1) \qquad\qquad -\infty \leq l_i \leq x_i \leq u_i \leq +\infty \qquad (2)$$

- for some $i \in B$ variable x_i is assigned $v(x_i) \notin \mathbb{Z}$
- for all $j \in N$ value $v(x_j)$ is l_j or u_j

Notation

- write $c = v(x_i) - \lfloor v(x_i) \rfloor$
- by assumption all nonbasic variables are assigned bounds, so we can split

$$L = \{\, j \in N \mid v(x_j) = l_j \,\} \qquad U = \{\, j \in N \mid v(x_j) = u_j \,\} \setminus L$$

$$L^+ = \{\, j \in L \mid A_{ij} \geq 0 \,\} \qquad U^+ = \{\, j \in U \mid A_{ij} \geq 0 \,\}$$

$$L^- = \{\, j \in L \mid A_{ij} < 0 \,\} \qquad U^- = \{\, j \in U \mid A_{ij} < 0 \,\}$$

Lemma (Gomory Cut)

cut is given by inequality

$$\frac{1}{1-c} \cdot \sum_{j \in L^+} A_{ij}(x_j - l_j) - \frac{1}{1-c} \cdot \sum_{j \in U^-} A_{ij}(u_j - x_j) - \frac{1}{c} \cdot \sum_{j \in L^-} A_{ij}(x_j - l_j) + \frac{1}{c} \cdot \sum_{j \in U^+} A_{ij}(u_j - x_j) \geq 1$$

Difference Logic

conjunction of constraints of the form $x - y \leqslant c$ or $x - y < c$

Definition Inequality Graph

conjunction φ of nonstrict difference constraints

- inequality graph of φ contains edge from $x \xrightarrow{c} y$ for every constraint $x - y \leqslant c$ in φ

Theorem

conjunction φ of nonstrict difference constraints is satisfiable $\iff$
inequality graph of φ has no negative cycle

Bellman-Ford Algorithm

computes distances in graphs from single source; detects negative cycles

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Consequence

Checking array bounds does not need special logic about arrays; integer arithmetic suffices

Example (Checking Array-Bounds)

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    - correct `while (i < N)` to `while (i + 1 < N)` in program

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  - array read (array index): $a[i]$ read array a at index i
  - array equality: $a = a'$ compare two arrays

Example (Setting up Verification Conditions)

- program for initializing an array with “true” ($\top$ in mathematical notation)

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  - suitable choice: Presburger arithmetic (linear arithmetic over $\mathbb{Z}$ with quantifiers)

Semantics of Array Logic (meaning of array-index, -update, -equality)

- array congruence: if arrays are equal and indices are equal, then identical elements are obtained when reading from an array

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- optional extensionality rule: two arrays are equal if they store the same elements

$$\forall a, b \in T_A.\ (\forall i \in T_I.\ a[i] = b[i]) \longrightarrow a = b \tag{3}$$

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  - arrays behave like uninterpreted functions: according to (1), reading an array at the same index yields same elements; function invocations on same inputs return same result
  - translation
    - for each array a introduce corresponding unary uninterpreted function A
    - array read access $a[i]$ is translated to function application $A(i)$

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- consider array logic formula with element type being characters

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- validity of formula can be shown by decision procedure for equality and uninterpreted functions (EUF)

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  whose validity is easily proven:
  apply equality $b[i] = e$ and prove resulting LIA constraint $e + 2 \geq e$

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Eliminating the Array Terms – Array Updates, continued

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- formula $a[0] = 5 \longrightarrow a\{7 \leftarrow x + 1\}\,[0] = 5$ is translated into

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Eliminating the Array Terms – Array Updates, continued

- translate $a\{i \leftarrow e\}$ via write rule:
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  whose validity can easily be proven in EUF + LIA after its translation

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Consider negated (simplified) verification condition from before; aim: show unsatisfiability

$$\underbrace{(\forall x \in \mathbb{Z}.\ x < i \longrightarrow a[x])}_{\text{precondition}} \wedge \underbrace{a' = a\{i \leftarrow \top\}}_{\text{loop iteration}} \wedge \underbrace{\neg(\forall x \in \mathbb{Z}.\ x < i + 1 \longrightarrow a'[x])}_{\text{negated postcondition}}$$

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- note that replacing $x < i$ by $x + 1 \leq i$ does not work, since $x + 1$ is no iterm; reason: x is universally quantified

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- remove existential quantifier: (eliminate $\exists x$ by fresh z)

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  - add $i-1$ because of $x \leq i-1$ and $j \leq i-1$
  - add $i+1$ because of $i+1 \leq j$
- replace universal quantifier:

$$\Big(\bigwedge_{x \in \mathcal{I}(\phi)} x \leq i-1 \longrightarrow a[x]\Big) \wedge a'[i] \wedge \Big(\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i-1 \vee i+1 \leq j \longrightarrow a'[j] = a[j]\Big) \wedge z \leq i \wedge \neg a'[z]$$

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$$(\forall x.\, x \leq i - 1 \longrightarrow a[x]) \wedge a' = a\{i \leftarrow \top\} \wedge \neg(\forall x.\, x \leq i \longrightarrow a'[x])$$

- formula after quantifier elimination, where $\mathcal{I}(\phi) = \{i, z, i-1, i+1\}$:

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- final formula: replace array read access by uninterpreted functions

$$\Big(\bigwedge_{x \in \mathcal{I}(\phi)} x \leq i - 1 \longrightarrow A(x)\Big) \land A'(i) \land \Big(\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i - 1 \lor i + 1 \leq j \longrightarrow A'(j) = A(j)\Big) \land z \leq i \land \neg A'(z)$$

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$$(\forall x.\ x \leq i - 1 \longrightarrow a[x]) \wedge a' = a\{i \leftarrow \top\} \wedge \neg(\forall x.\ x \leq i \longrightarrow a'[x])$$

- formula after quantifier elimination, where $\mathcal{I}(\phi) = \{i, z, i-1, i+1\}$:

$$(\bigwedge_{x \in \mathcal{I}(\phi)} x \leq i - 1 \longrightarrow a[x]) \wedge a'[i] \wedge (\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i - 1 \vee i + 1 \leq j \longrightarrow a'[j] = a[j]) \wedge z \leq i \wedge \neg a'[z]$$

- final formula: replace array read access by uninterpreted functions

$$(\bigwedge_{x \in \mathcal{I}(\phi)} x \leq i - 1 \longrightarrow A(x)) \wedge A'(i) \wedge (\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i - 1 \vee i + 1 \leq j \longrightarrow A'(j) = A(j)) \wedge z \leq i \wedge \neg A'(z)$$

- unsatisfiability now decidable: consider cases $z \leq i - 1 \vee z = i \vee z \geq i + 1$ via LIA reasoning

Example Reduction Algorithm, completed

- input:

$$(\forall x.\ x \leq i - 1 \longrightarrow a[x]) \wedge a' = a\{i \leftarrow \top\} \wedge \neg(\forall x.\ x \leq i \longrightarrow a'[x])$$

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- unsatisfiability now decidable: consider cases $z \leq i - 1 \vee z = i \vee z \geq i + 1$ via LIA reasoning
  - case $z = i$: show unsatisfiability using $A'(i)$ and $\neg A'(z)$ via EUF

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  - case $z = i$: show unsatisfiability using $A'(i)$ and $\neg A'(z)$ via EUF
  - case $z \leq i-1$: since $z \in \mathcal{I}(\phi)$, obtain $A(z)$, $A'(z) = A(z)$, and $\neg A'(z)$ and use EUF

Example Reduction Algorithm, completed

- input:

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- unsatisfiability now decidable: consider cases $z \leq i-1 \vee z = i \vee z \geq i+1$ via LIA reasoning
  - case $z = i$: show unsatisfiability using $A'(i)$ and $\neg A'(z)$ via EUF
  - case $z \leq i-1$: since $z \in \mathcal{I}(\phi)$, obtain $A(z)$, $A'(z) = A(z)$, and $\neg A'(z)$ and use EUF
  - case $z \geq i+1$: show unsatisfiability in combination with $z \leq i$ via LIA

Theorem (Correctness of Reduction Algorithm)

The input formula and the result of the reduction algorithm are equisatisfiable.

Corollary

If satisfiability of quantifier-free $T_{EUF} \cup T_{LIA} \cup T_E$ formulas is decidable, then so is satisfiability of the fragment of array logic for T_E.

A Problem and its Solution

- in the reduction algorithm, the universal part of the write rule

$$\forall j.\, j \leq i - 1 \vee i + 1 \leq j \longrightarrow a'[j] = a[j]$$

  is turned into a finite conjunction

$$\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i - 1 \vee i + 1 \leq j \longrightarrow a'[j] = a[j]$$

- problem: this formula often gets (too) large

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- observation: implications are often only required for a few index terms within $\mathcal{I}(\phi)$ (in previous example, only the index term z was required to prove unsatisfiability)

A Problem and its Solution

- in the reduction algorithm, the universal part of the write rule

$$\forall j.\ j \leq i - 1 \lor i + 1 \leq j \longrightarrow a'[j] = a[j]$$

  is turned into a finite conjunction

$$\bigwedge_{j \in \mathcal{I}(\phi)} j \leq i - 1 \lor i + 1 \leq j \longrightarrow a'[j] = a[j]$$

- problem: this formula often gets (too) large
- observation: implications are often only required for a few index terms within $\mathcal{I}(\phi)$ (in previous example, only the index term z was required to prove unsatisfiability)
- solution: use a lazy encoding procedure, that generates instances only on demand, and can be combined with DPLL(T)
- details: see literature, in particular Section 7.4 of Decision Procedures book

Outline

Summary

- checking array bounds is easily encoded via LIA, does not require extension of logic
- array logic provides primitives for array read- and write-accesses
- arrays are easily modeled as uninterpreted functions
- array logic is often undecidable, even for decidable index- and element-theories such as Presburger arithmetic
- array properties define a fragment of array logic;
  the fragment can be translated to quantifier-free formulas by adding EUF
- optimization: lazy encoding creates instances of the write rule on demand

Kröning and Strichmann

- Sections 7.1–7.3
- warning: mistake in example at end of Section 7.3 (over-simplification)
  - problem: $<$ not eliminated
  - result of mistake: smaller set of index terms $\mathcal{I}(\phi) = \{i, z\}$, but correct set is $\{i, i-1, z\}$
  - incorrect set does not cause problems in example, but in general elimination of $<$ is essential

Further Reading

Aaron R. Bradley, Zohar Manna, Henny B. Sipma
What's Decidable About Arrays?
Proc. VMCAI 2006, volume 3855 of LNCS, pages 427-–442, 2006

Important Concepts

- array logic
- array property
- checking array bounds via LIA
- invariants
- reduction algorithm
- spurious counterexample
- write rule