Discrete Structures/Mathematics WS 2021/2022 LVA 703069

EXAM 1 1 February 2022

> This exam consists of <u>three</u> regular exercises (1–3) worth 70 points in total. The available points for each item are written in the margin. You need at least 30 points to pass. Always explain your answer. In particular, for yes/no questions the correct answer is worth 1 point with the remaining points for the explanation. The time available is 1 hour and 45 minutes (105 minutes).

Throughout this exam, let the words f and l be your first respectively last name written (in lowercase, omitting diacritics) over the alphabet $\Sigma = \{\texttt{a}, ..., \texttt{z}\}$, and let m be your Matrikelnr. having 8 digits $m_1m_2m_3m_4m_5m_6m_7m_8$ with $0 \leq m_i \leq 9$. **Start each document handed in with (writing down) your f, l, and m.**

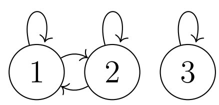
1 (a) Let Σ be a finite alphabet and let $A \subseteq \Sigma^*$ be a language. What are the
[4] possible values of $|A^*|$?

Solution: If $A = \emptyset$ or $A = \{\varepsilon\}$ then $A^* = \{\varepsilon\}$ and hence $|A^*| = 1$. Otherwise, we have $|A| = |\mathbb{N}|$ since on one hand $A \subseteq \Sigma^*$ is countable and on the other there exists a $w \in A \setminus \{\varepsilon\}$ so that $n \mapsto w^n$ is an injection from $\mathbb{N}$ to A, which implies $|A| \geq |\mathbb{N}|$.

Here, ε stands for the empty word. Note that ∞ is *not* a sufficient answer since we asked for the *cardinality*, and there are many different infinite cardinalities.

(b) True or false: if R is an equivalence relation over a finite set A, all
[3] equivalence classes of R have the same number of elements.

Solution: False. The equivalence relation on the right has the classes $\{1, 2\}$ and $\{3\}$.



(c) True or false: If a set $A \subseteq \{0, 1\}^*$ is recursive, then all subsets of A are
[3] also recursive.

Solution: False. $\{0, 1\}^*$ is trivially recursive, so if the statement were true, there would be no non-recursive sets. But we have seen in the lecture that non-recursive languages exist.

(d) True or false: every non-recursive set $A \subseteq \{0, 1\}^*$ has a non-recursive
[4] proper subset.

Solution: True. Since the empty set is recursive, we know that there exists some $x \in A$. The proper subset $A \setminus \{x\}$ is then also non-recursive, since if it were recursive, then $A \setminus \{x\} \cup \{x\} = A$ would also be recursive ($\{x\}$ is recursive, and the union of two recursive sets is recursive).

(e) Recall that the halting problem $HP = \{M\#x \mid M$ halts for input $x\}$ is not recursive. Argue that $UHP = \{M \mid M$ halts for all inputs$\}$ is then also non-recursive. [6]

Solution: If *UHP* were recursive, we could decide *HP* by constructing a Turing machine M' that deletes its input, writes x onto the tape, and then runs M, and then checking whether $M' \in UHP$. Clearly, $M' \in UHP \iff M\#x \in HP$.

More formally, one can also prove the reduction $HP \leq UHP$ using a reduction function that maps $M\#x$ to the machine M' above (and every string that is not of the form $M\#x$ to a machine that loops for all inputs). This then also shows that *UHP* is not recursive.

(f) True or false: if $A_0, A_1, \ldots$ are a countable family of recursive languages $\subseteq \{0,1\}^*$, then $\bigcup_{i\in\mathbb{N}} A_i$ is also recursive. [3]

Solution: False. Every subset of $\{0,1\}^*$ is countable and can thus be written as a countable union of single-element languages (which are obviously recursive). If the union of these were also recursive, all subsets of $\{0,1\}^*$ would be recursive. But in the lecture, we have seen that there are non-recursive languages.

(g) True or false: Let G be a finite undirected graph and let e be some edge in G. Then there exists a spanning forest of G that contains e. [4]

Solution: The answer depends on whether or not G can contain loops. According to the lecture, loops were allowed. However, for the exam, both interpretations were accepted as correct.

If G can contain loops, the statement is false, since clearly a loop cannot be part of a spanning forest.

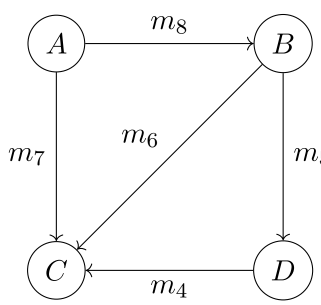
If G cannot contain loops, the statement is true. We assign the weight 0 to e and the weight 1 to all other edges. Running Kruskal's algorithm will give us a spanning forest of G that clearly contains e, since e is added in the very first step.

(h) Let Σ be a finite alphabet and n be a natural number. How many palindromic words from Σ^n are there (i.e. reversing the string does not change it, e.g. *abccba*). [3]

Solution: $|\Sigma|^{n/2}$ if n is even or $|\Sigma|^{(n+1)/2}$ if n is odd. This is because we have free choice of letters for the first half of the word (including the middle one if n is odd) but the remaining ones are then fixed.

More formally: we have a bijection between $\Sigma^{\lceil n/2 \rceil}$ and the set of palindromes over Σ of length n given by appending the first $\lfloor n/2 \rfloor$ characters at the end in one direction, and throwing away the last $\lfloor n/2 \rfloor$ letters in the other direction.

2 (a) Use Floyd's algorithm to compute the shortest paths among the nodes of the following weighted digraph G. Give the start matrix and all intermediate matrices. Enumerate the nodes alphabetically, that is, the first row in a matrix corresponds to node A, second to B, and so on. [9]



Solution: The start matrix would be N_0 The shortest distance matrix would be N_4 where $x_1 = \mathsf{min}(m_7, m_8+m_6, m_8+m_5+m_4)$ and $x_2 = \mathsf{min}(m_6, m_5+m_4)$. In matrix N_2 and N_3 we have $x_3 = \mathsf{min}(m_7, m_8+m_6)$.

$$N_0 = N_1 = \begin{pmatrix} 0 & m_8 & m_7 & \infty \\ \infty & 0 & m_6 & m_5 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & m_4 & 0 \end{pmatrix} \quad N_2 = N_3 = \begin{pmatrix} 0 & m_8 & x_3 & (m_8+m_5) \\ \infty & 0 & m_6 & m_5 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & m_4 & 0 \end{pmatrix}$$

$$N_4 = \begin{pmatrix} 0 & m_8 & x_1 & (m_8+m_5) \\ \infty & 0 & x_2 & m_5 \\ \infty & \infty & 0 & \infty \\ \infty & \infty & m_4 & 0 \end{pmatrix}$$

[2] (b) Give the transitive closure G^+ of G.

Solution: $G = \{(A,B),(A,C),(B,C),(B,D),(D,C)\}$

$$G^+ = \{(A,B),(A,C),(B,C),(B,D),(D,C),(A,D)\}$$

(c) Let H be multigraph G where the edges are undirected. Compute a minimal spanning tree of H using Kruskal's algorithm. Show all intermediate steps and the final spanning tree. [9]

Solution: For the computation, we use the example Matrikelnr. $m = 12345678$. Kruskal's algorithm starts with $F = \varnothing$; $P = \{\{A\},\{B\},\{C\},\{D\}\}$.

0	$F = \varnothing$	$P = \{\{A\},\{B\},\{C\},\{D\}\}$
1	$F = \{\{C,D\}\}$	$P = \{\{A\},\{B\},\{C,D\}\}$
2	$F = \{\{C,D\},\{B,D\}\}$	$P = \{\{A\},\{B,C,D\}\}$
3	$F = \{\{C,D\},\{B,D\},\{A,C\}\}$	$P = \{\{A,B,C,D\}\}$

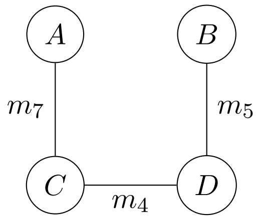


Figure 1: Minimal spanning tree of H.

3 Let M be the set of all numbers in your m, let F be the set of all characters in your f, and let L be the set of all characters in your l. When the solution is a large number, keep it as an expression (for example, like $(10!)^{3+9}$). Always explain how you obtained your solution.

[1] (a) i. Compute $\#(M)$, $\#(F)$, and $\#(L)$.

[2] ii. How many subsets of L are there?

Solution: $2^{(\#L)}$ since every subset can be seen as a characteristic function from L to $\{0, 1\}$, or as a binary number with $\#L$ bits.

[2] iii. How many matrices of the shape $\#(F) \times \#(L)$ with elements from M are there?

Solution: $(\#M)^{(\#F \cdot \#L)}$ since we have $\#M$ choices for each of the $\#F \cdot \#L$ elements.

[3] iv. How many injective functions from F to L are there?

Solution: If $\#F > \#L$ then there are none. Otherwise the count is the falling factorial $\#L \cdot (\#L - 1) \cdot \ldots \cdot (\#L - \#F + 1)$.

[3] v. How many symmetric relations on M are there?

Solution: $2^{(\#M \cdot (\#M+1)/2)}$ since every relation is uniquely determined by a matrix of the shape $\#M \times \#M$ with 2 elements (say 0 and 1). Symmetric relations are represented by symmetric matrices, hence we have 2 choices for every element below and on the diagonal (that is, we can choose 1 element in the first row, 2 elements in the second row, ...).

(b) In the following consider *directed* graphs (without loops and multi-edges).

[3] i. How many graphs with nodes M are there?

Solution: $2^{(\#M \cdot (\#M-1))}$ since every graph on $\#M$ nodes can be uniquely determined by a matrix of the shape $\#M \times \#M$ with 2 elements (say 0 for no edge, and 1 for an edge). Since there are no loops, the matrix must have zeros on the diagonal and hence there is $(\#M - 1)$ in the exponent.

[3] ii. How many graphs with nodes M with edges labeled by F are there?

Solution: $(\#F + 1)^{(\#M \cdot (\#M-1))}$ similarly to the above. Here we, however, represent the graph by a matrix (with zeros on the diagonal) which contains one of the $\#F$ weight values as the elements. And additionally, we need one more value when there is no edge.

[3] (c) True or False: An undirected tree with a node of degree k contains at least k nodes of degree 1. Prove the claim or provide a counterexample.

Solution: The statement is true for finite tress and false for infinite tress. For infinite tress, we can easily construct a tree with infinite branches and no leaves. For finite tress, we prove the claim as follows.

First, let us prove that every tree T with at least 2 nodes has at least 2 nodes of degree 1 (leaves). Let $P = (v_0, e_1, v_1, \ldots, e_k, v_k)$ be a path in T

of a maximal possible length. It is easy to see that both v_0 and v_k have degree 1 because otherwise the path can be extended (which contradicts the maximality of P) or there is a cycle (which contradicts that T is a tree).

Now let a tree T contain a node v of degree k. By removing v from T we obtain a forest with k trees $T_1, \ldots, T_k$. Let v_i be the node in T_i where T_i was connected to v. When T_i is a single node tree, then v_i has degree 0 in T_i and hence v_i has degree 1 in T. When T_i has more than 2 nodes, then T_i has at least two nodes of degree 1 (in T_i) as proved above. One of these two nodes is not v_i and thus it has degree 1 in T as well. Hence we have found at least k nodes of degree 1 (one in every T_i). $\square$