universität innsbruck

Discrete Structures	WS 2021/2022	LVA 703069

EXAM 2 February 28, 2022

This exam consists of three regular exercises (1–3) worth 70 points in total. The available points for each item are written in the margin. You need at least 30 points to pass. Always explain your answer. In particular, for yes/no questions the correct answer is worth 1 point with the remaining points for the explanation. The time available is 1 hour and 45 minutes (105 minutes).

Throughout this exam, let the words f and l be your first respectively last name written (in lowercase, omitting diacritics) over the alphabet $\Sigma = \{\mathtt{a}, \dots, \mathtt{z}\}$, and let m be your Matrikelnr. having 8 digits $m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8$ with $0 \leq m_i \leq 9$. **Start each document handed in with (writing down) your f, l, and m.**

1 (a) True or false: all equivalence relations over a finite set A have the same number of classes. [3]

Solution: False. If a set A has n elements, $A \times A$ has 1 class (everything is equivalent) and $\{(x, x) \mid x \in A\}$ has n classes (no two different elements are equivalent).

(b) Let A be an arbitrary set. Give all relations on A that are both partial orders and equivalence relations. [3]

Solution: Partial orders are antisymmetric. Equivalence relations are symmetric. Therefore, the relation cannot contain any pair (x, y) with $x \neq y$. Since equivalence relations must additionally be reflexive, the only such relation is therefore the diagonal relation $\{(x, x) \mid x \in A\}$.

(c) True or false: every language is either recursive or recursively enumerable. [3]

Solution: False, since recursive implies semi-recursive and there are sets that are not recursively enumerable.

[3] (d) True or false: If A and B are non-recursive, then $A \cap B$ is non-recursive.

Solution: False. Let A be any non-recursive language and B its complement. Then $A \cap B = \emptyset$, which is trivially recursive.

(e) From the lecture, you know the halting problem HP. Consider the following bounded version of it:

$$\text{BHP} := \{M\#n\#x \mid M \text{ uses no more than } n \text{ tape for input } x\}$$

where M is a Turing machine, n is a natural number, and x is a word. Assume that M, n, and x are encoded as bit strings.

[5] "Uses no more than n tape" means that the read-write head of M never moves more than n tape cells away from its initial position. Is L recursive?

Solution: Yes. We can simply simulate the machine until it either moves outside the allowed tape region, or it halts, or it reaches a configuration that we have seen already. Since there are only finitely many configurations (i.e. tuples of the current state, tape content, and head position) that stay within the allowed tape region, one of the three options must happen.

(f) Recall that the halting problem $HP = \{M\#x \mid M \text{ halts for input } x\}$ is not recursive. Argue that

$$A = \{M \mid \exists x \in \{0,1\}^*.\ M \text{ halts for input } x \text{ in at most } |x| \text{ steps}\}$$

[5] is then also non-recursive.

Solution: If A were recursive, we could decide HP with it: given a machine M and an input word x, we modify M to obtain a new machine M' by adding a few states at the beginning that erase the tape and write x onto it. In other words, M' ignores its input and behaves the same as M running on input x.

Thus, M' terminates in at most $|y|$ steps on input y if and only if M terminates in at most $|y|$ steps on input x. And thus, $M' \in A$ if and only if M terminates on input x, i.e. if $M\#x \in HP$.

Alternative proof: one can also prove the reduction $HP \leq A$ using a reduction function that maps $M\#x$ to the machine M' above (and every string that is not of the form $M\#x$ to a machine that loops for all inputs). This then also shows that A is not recursive.

(g) For what sets A does $\emptyset \leq A$ hold (where $\leq$ denotes computable re-
[4] ducibility).

Solution: A reduction function f must satisfy $\forall w.\ f(w) \in A \iff w \in \emptyset$, i.e. in our case every input word must be mapped to something not in A. This is clearly possible if and only if $A \subsetneq \Sigma^*$. A possible reduction function is $f(x) = c$ (where c is an arbitrary element in $\Sigma^* \setminus A$).

(h) Let Σ be a finite alphabet and $n > 0$ be a natural number. How many words from Σ^n contain the same letter in adjacent positions somewhere?
[4] (e.g. *aabc*, but not *abac*)

Solution: Let $k = |\Sigma|$. There are $k(k-1)^{n-1}$ words that do *not* contain such a pattern (first letter can be anything, every successive letter can only be chosen in $k-1$ ways since it must not equal the previous one). Thus, the desired number is $k^n - k(k-1)^{n-1}$

2 (a) Consider the relation

$$R = \{(m_1, m_3), (m_3, m_2), (m_4, m_5), (m_5, m_3)\}$$

[2] i. Is R anti-symmetric? Explain your answer.

Solution: R is anti-symmetric, if for all $m_i, m_j \in M$: $(m_i, m_j) \in R$ and $(m_j, m_i) \in R \implies m_i = m_j$.
In particular in our example R is not anti-symmetric if one of the following cases are satisfied.

- $m1 = m2$ but $m1 \neq m3$
- $m3 = m4$ but $m3 \neq m5$
- $m2 = m5$ but $m2 \neq m3$
- $m1 = m5 \wedge m3 = m4$ but $m1 \neq m3$
- $m3 = m5 \wedge m2 = m4$ but $m3 \neq m2$

Otherwise R would be anti-symmetric.
Hence we know that the example Matrikelnr $m = 12345678$ is anti-symmetric.

[2] ii. Is R irreflexive? Explain your answer.

Solution: R is irreflexive, if for all $x \in M$, $(x, x) \notin R$. In particular in our example R is not irreflexive if one of the following cases are satisfied.

- $m1 = m3$
- $m3 = m2$
- $m4 = m5$
- $m5 = m3$

Otherwise R would be irreflexive.
Hence we know that the example Matrikelnr $m = 12345678$ is irreflexive.

[4] iii. Compute the reflexive-transitive closure R^* of R and explain how you did that.

Solution: $R^* = \{(1,3), (3,2), (4,5), (5,3), (1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (4,3), (4,2), (5,2)\}$ is the reflexive-transitive closure of the example Matrikelnr $m = 12345678$.

(b) Let $a = (m_1 \bmod 2) + 2$, where mod is the modulo operation. Determine for each of the following functions which properties (injective, surjective, bijective) they have or don't have.

[2] i. $f_1 : \mathbb{N} \to \mathbb{N}, \quad f_1(x) = x^a$

[2] ii. $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(x) = x^a$

[2] iii. $f_3 : \mathbb{R} \to \mathbb{R}_{\geq 0}, f_3(x) = |ax|$

Solution: There are two possibilities for a. Either $a = 2$ or $a = 3$.
If $a = 2$

i. injective, not surjective, not bijective

ii. not injective, not surjective, not bijective

iii. not injective, surjective, not bijective

If $a = 3$

i. injective, not surjective, not bijective

ii. injective, surjective, bijective

iii. not injective, surjective, not bijective

(c) Let $f_4 : \mathbb{Z}^2 \to \mathbb{Z}, f_4(x, y) = m_5 x + y$. For each property (injective, surjective, bijective) show that f_4 has the property or provide a counter example if not. [6]

Solution:

- f_4 is not injective because $f_4(0, m_5) = f_4(1, 0)$.
- f_4 is surjective. We can set $x = 0$. Then $f_4(0, y) = y$.
- f_4 is not bijective because it was not injective.

3 Let M be the set of all numbers in your m, let F be the set of all characters in your f, and let L be the set of all characters in your l.

[1] (a) i. Compute $\#(M)$, $\#(F)$, and $\#(L)$.

[2] ii. How many relations on M are there?

Solution: We have $(\#M)^2$ pairs of elements from M. Every subset of these pairs is one relation, so there are $2^{(\#M)^2}$ relations in total.

[2] iii. How many subsets of L of size k are there (k is arbitrary)?

Solution: This is the binomial coefficient $\binom{\#L}{k}$.

iv. A *partial function* from A to B is a subset of $A \times B$ which satisfies the uniqueness functional property but is not necessarily total. How many partial functions from F to L are there? [3]

Solution: Every partial function f can be made total by assigning the unassigned values to a special value not found in L (say $\bullet \notin L$). The total function f' is the completion of f defined as follows:

$$f'(x) = \begin{cases} f(x) & \text{if } f(x) \text{ is defined} \\ \bullet & \text{otherwise} \end{cases}$$

This gives us a bijection between the partial functions from F to L and the total functions from F to $L \cup \{\bullet\}$. Hence the number of the partial functions is $(\#L + 1)^F$.

[3] v. How many reflexive relations on M are there?

Solution: Every relation on M can be uniquely described by a matrix of shape $\#M \times \#M$ with two values (say 0 and 1). A reflexive relation must have all 1 on the diagonal ($\#M$ values), but we are free to choose all the other elements. Hence the solution is $2^{\#M(\#M-1)}$.

(b) In the following consider *undirected* graphs (without loops).

[3] i. How many graphs with nodes M are there?

Solution: There are $\binom{\#M}{2}$ undirected edges and because every subset of these edges describes exactly one graph we have $2^{\binom{\#M}{2}}$ graphs.

ii. How many graphs with nodes M with edges labeled by F are there?

[3]

Solution: Every labeled graph can be seen as labeling function from the set of edges to the labels with one additional label for edges not present in the graph. Hence the solution is $(\#F+1)^{\binom{\#M}{2}}$.

(c) True or False: A finite undirected graph (without loops) with n nodes and c components contains at least $n-c$ edges. Prove the claim or provide a counterexample.

[3]

Solution: True. Let us consider a graph $G=(V,E)$ with n nodes and c components with the minimal possible count of edges. This must be a forrest since otherwise we can remove an edge while keeping the number of components c. From the lecture/proseminar we know that a forrest with c trees and n nodes has exactly $n-c$ edges. Since G was minimal with respect to the count of edges, every other graph with n nodes and c components will have at least $n-c$ edges. □