Discrete Structures/Mathematics WS 2021/2022 LVA 703069

EXAM 3 30 September, 2022

This exam consists of <u>three</u> regular exercises (1–3) worth 70 points in total. The available points for each item are written in the margin. You need at least 30 points to pass. Always explain your answer. In particular, for yes/no questions the correct answer is worth 1 point with the remaining points for the explanation. The time available is 1 hour and 45 minutes (105 minutes).

Throughout this exam, let the words f and l be your first respectively last name written (in lowercase, omitting diacritics) over the alphabet $\Sigma = \{\mathtt{a}, \ldots, \mathtt{z}\}$, and let m be your Matrikelnr. having 8 digits $m_1m_2m_3m_4m_5m_6m_7m_8$ with $0 \leq m_i \leq 9$. **Start each document handed in with (writing down) your f, l, and m.**

1

[4] (a) True or false: an equivalence relation $R \subseteq A \times A$ has at most $|A|$ equivalence classes.

Solution: True. The union of all the classes is A and the classes are distinct. Since every class contributes at least one element to the union (as they are non-empty), the cardinality of the union has to be at least as big as the cardinality of the sets of all classes.

More formally: Let us pick one representative for each equivalence class C and call it $f(C)$. Then f is an injection from the set of equivalence classes of R into A, so the number of classes is $\leq |A|$.

[3] (b) True or false: Let A be a finite set and let $R \subseteq A \times A$ be a strict partial order. Then R is well-founded.

Solution: True. By the pigeonhole principle, any infinite descending chain would have to contain the same element twice, i.e. there would be a subchain $xR \ldots Rx$. By transitivity, we have $(x, x) \in R$, which contradicts the irreflexivity of R.

[3] (c) True or false: if $A_1, \ldots, A_n$ are recursive languages, then $A_1 \cup \ldots \cup A_n$ are also recursive.

Solution: True. We can simply take the n Turing machines for the languages A_i, run them after another, and output “True” as soon as one of them says “True” and “False” otherwise.

[3] (d) True or false: Let $A, B \subseteq \{0, 1\}^*$ be languages with $A \leq B$ and $B \leq A$ (where $\leq$ denotes computable reducibility). Then $A = B$.

Solution: False. Any two recursive sets (other than $\emptyset$ and Σ^*) are reducible to one another.

(e) Recall that the halting problem $HP = \{M\#x \mid M \text{ halts for input } x\}$ is not recursive. Argue that

$$EqHP = \{M_1\#M_2 \mid M_1 \text{ and } M_2 \text{ halt on exactly the same inputs}\}$$

[6] is then also non-recursive.

Solution: Given M and x, let M' be the machine that halts immediately if its input is x and otherwise simulates M. Clearly M' has the same halting behaviour as M if and only if M halts on x. Thus, deciding $EqHP$ for M' also decides HP for M and x.

Alternatively, one can also do a reduction proof by using the mapping of $M\#x$ to $M\#M'$ (with M' defined as above) as the reduction function.

(f) For what sets A does $A \leq \emptyset$ hold (where $\leq$ denotes computable reducibility). Give the corresponding reduction functions. [3]

Solution: We must fulfil $\forall x.\ f(x) \in \emptyset \iff x \in A$, which simplifies to $\forall x.\ x \notin A$, which simplifies to $A = \emptyset$. So only $A = \emptyset$ works, and every function $f : \Sigma^* \to \Sigma^*$ is a reduction function.

(g) True or false: Let G be a finite undirected graph. Then all spanning forests of G have the same number of edges. [4]

Solution: True. All spanning forests of G clearly must have exactly the same nodes as G and the same connected components as G (otherwise they would not be spanning). And in a forest, "#edges = #nodes - #component", as we have seen in the lecture.

(h) How many numbers between 1 and 6000 are a multiple of at least one of the numbers 2, 3, and 5? [4]

Solution: We apply the principle of inclusion and exclusion:

- 3000 multiples of 2, 2000 multiples of 3, 1200 multiples of 5
- 1000 multiples of $2 \cdot 3$, 600 multiples of $2 \cdot 5$, 400 multiples of $3 \cdot 5$
- 200 multiples of $2 \cdot 3 \cdot 5$

Thus the desired number is $3000 + 2000 + 1200 - 1000 - 600 - 400 + 200 = 4400$.

2 (a) In an algorithm, we divide large problems into $r = (m_3 \bmod 4) + 2$ equal parts and discard $r - 1$ of them in constant time, then call the algorithm recursively on the remaining part. What is the complexity of this algorithm for size $n = r^k$ for $k \geq 0$. [6]

Solution: Our recurrence is $T(n) = T(\frac{n}{r}) + c$. The second case of the Master theorem applies since $a = 1 = r^0 = b^s$. Thus we have $T(n) \in \Theta(\log n)$ regardless of the precise value of r.

(b) Let $k_1 = 3m_7m_8$ in decimal notation (so $300 \leq k \leq 399$).

Determine *all* $0 \le x \le k$ that satisfy the 3 congruences:

$$\begin{aligned} x &\equiv 0 \pmod 3 \\ x &\equiv 2 \pmod 5 \\ x &\equiv 4 \pmod 7 \end{aligned}$$

[10] by application(s) of the Chinese Remainder Theorem, and check that your solutions for x satisfy the congruences. Give all computation steps and explain how you conclude that your solutions are the only ones.

(c) Let $k_2 = m_2m_3$ in decimal notation (so $0 \le k \le 99$). Compute

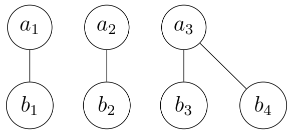
$$(k_2^{1998} + k_2) \pmod{1999}$$

given that 1999 is prime. State what theorem you used and what con-
[4] ditions must hold to apply it.

3 Let $k = (m_3 \bmod 3) + 1$ and $l = (m_4 \bmod 3) + 1$.

(a) Draw the Hasse diagram of a partial order that has k minimal elements
[2] and l maximal elements and no other elements.

Solution: Possible example for $k = 4$ and $l = 3$:



[3] (b) Is this the only partial order that satisfies this specification?

Solution: If $k = l = 1$: no, since both the empty relation with universe $\{0\}$ works, but so does the relation where $0 < 1$ on the universe $\{0, 1\}$.

If $k = 1$ and $l \neq 1$: yes, since all maxima have to be connected to the (unique) minimum and no other connections are possible (since otherwise the maxima would not be maximal). Analogously if $l = 1$ and $k \neq 1$.

Otherwise no, since one can e.g. connect each minimum either to one maximum or to two different ones.

Note: The intention in this question was that isomorphic partial orders (i.e. those that differ only by renaming the elements) are to be considered equal. Some students understandably interpreted it differently, such that e.g. the relations $\{(0, 1)\}^=$ and $\{(a, b)\}^=$ are different. Such solutions received full points as well.

[3] (c) Is your partial order a total order?

Solution: Only if $k = 1$ and $l = 1$, since a total order cannot have more than 1 minimal and maximal element.

[3] (d) Is it well-founded (as a partial order)?

Solution: By definition, a relation is well-founded as a partial order if its strict part is well-founded. This is the case in our example here, since the underlying set is finite.

(e) Let A be the set of partial orders over $\mathbb{N}$ and let $B = 2^{\mathbb{N}} \setminus \{\emptyset, \mathbb{N}\}$ be the set of all nontrivial subsets of $\mathbb{N}$ (i.e. all subsets that are not $\emptyset$ or $\mathbb{N}$). Show that $|A| \geq |B|$ by giving an injection $B \to A$.

[6] (You need not prove formally that it is an injection, but you do have to explain it.)

Hint: take the relation you gave in a) as an inspiration.

Solution: We construct an injection $B \to A$ in the following way: given a set $X \subseteq \mathbb{N}$, we return the reflexive closure of $X \times (\mathbb{N} \setminus X)$. That is, any element in X is strictly less than any element in $\mathbb{N} \setminus X$. The minimal elements of this order are precisely X and the maximal elements are $\mathbb{N} \setminus X$, which also shows that this is indeed an injection.

[3] (f) Argue why the previous subexercise implies that $|A| > |\mathbb{N}|$.

Solution: $2^{\mathbb{N}}$ is clearly infinite, and removing finitely many elements (in this case two) from an infinite set does not change its cardinality. Hence $|B| = |2^{\mathbb{N}}|$. In the lecture we have seen that $2^{\mathbb{N}}$ is uncountably infinite, while $\mathbb{N}$ is countably infinite, hence $|2^{\mathbb{N}}| > |\mathbb{N}|$.