1) Show that for any infinite set A and any $x \in A$, we have $|A \setminus \{x\}| = |A|$.
Hint: Note that any infinite set has a countably infinite subset. Try proving that the statement holds for all countably infinite sets A first.

2) a) Show that the set of rational numbers $\mathbb{Q}$ is countably infinite.
   b) Let $\mathcal{G}$ be the set of graphs $G = (V, E)$ with $V \subseteq \mathbb{N}$. Show that $\mathcal{G}$ is countably infinite.
   c) Show that $|[0,1]| = |(0,1)|$
   d) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Show that $|[a,b]| = |[c,d]|$.
   e) Show that $|[0,1]| = |\mathbb{R}|$.

Hint: Recall that there are two basic ways of showing that two sets have the same cardinality:

- directly, i.e. by giving a bijection between them, or
- through the Schröder–Bernstein theorem, i.e. showing $|A| \leq |B|$ and $|B| \leq |A|$ by giving injective functions $f : A \to B$ and $g : B \to A$

Recall that $[a,b] = \{x \mid x \in \mathbb{R},\ a \leq x \leq b\}$ and $(a,b) = \{x \mid x \in \mathbb{R},\ a < x < b\}$.

3) State if the following statements are true or false. Justify your answer with a proof:
   a) $\mathbb{R}$ is countable.
   b) $\mathbb{N} \times \mathbb{Q}$ is countable.
   c) If A is countable but B is not, then $A \cap B$ is countable.
   d) If A is a infinite set, then A^* is countable.

4*) Suppose that in an infinitely near future, the Discrete Structure course becomes so popular that countably infinitely many students want to attend it. A new lecture room is built, with countably infinitely many seats numbered by natural numbers. At the first lecture, all the seats are taken but one student arrives late and has no place to sit. The lecturer asks everyone to stand up and move from seat n to seat $n + 1$. This makes the seat number 0 unoccupied and the late student can sit there. Now, you – the best student from this year – are asked to take care of other late students, and find some place for them to sit. How would you handle the following situations?

   a) 10 more students arrive late. How will you move the present students, and where will the i-th late student sit?
   b) A bus with countably infinitely many students arrives late. How will you move the students, and where will the i-th student from the bus sit?
   c) Countably infinitely many buses, each with countably infinitely many students, arrive late. How will you move the present students, and where will you place student i from bus j? Don't worry if some places are left unoccupied afterwards.