1) Specify a Turing machine that checks for balanced brackets. We only consider the brackets "{" and "}". The Turing machine should accept if the brackets are balanced e.g. {{}{}}{} or reject otherwise e.g. {{}.

2) Given a partial order $\leq$ on a set A we say that $x \in A$ is a minimal element of $\leq$ if there is no $y \in A$ such that $y < x$. Analogously, $x \in A$ is a maximal element if there is no $y \in A$ such that $y > x$.

   List the minimal and maximal elements of the following partial orders:

   i) the natural order ($\leq$) on $\mathbb{N}$

   ii) the divisibility order ($|$) on $\mathbb{N}$

   iii) the divisibility order ($|$) on $\mathbb{N} \setminus \{0, 1\}$

   iv) the subset order ($\subseteq$) on the subsets of $\{1, 2, 3, 4\}$ that do not contain both 2 and 4

   Hint: It may help to sketch the graph of the relations.

3) Recall that a function $f : A \to B$ is

   - "injective" if $\forall x, y{\in}A.\ f(x) = f(y) \longrightarrow x = y$
   - "surjective" if $\forall y{\in}B.\ \exists x{\in}A.\ f(x) = y$
   - "bijective" if it is injective and surjective

   a) For a function $f : \mathbb{R} \to \mathbb{R}$, what does "injective" and "surjective" mean geometrically? How can you tell if f is injective/surjective by looking at the graph of f?

   b) Determine for each of the following functions which of these three properties they have:

   i) $f_1 : \mathbb{N} \to \mathbb{N},\ f_1(x) = x^2$
   ii) $f_2 : \mathbb{Z} \to \mathbb{Z},\ f_2(x) = x^2$
   iii) $f_3 : \mathbb{N} \to \mathbb{N},\ f_3(x) = x^3$
   iv) $f_4 : \mathbb{R} \to \mathbb{R}_{\geq 0},\ f_4(x) = x^2$
   v) $f_5 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ f_5(x) = x^2$

   Note: $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

   c) Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z},\ \ f(a, b) = a - b$ is surjective.

   d) Show that the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}_{>0},\ \ f(a, b) = (2a + 1)2^b$ is bijective.

4*) Show that the following modified notions of Turing Machines are equivalent to the ones you know from the lecture:

   a) Turing Machines that can also make transitions without moving their read/write head.

   Formally: the transition function now has the form

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, N\}$$

   with the transition function extended in the following way:

$$(p, y\sqcup^\infty, n) \to (q, y_0 \dots b \dots y_{m-1}\sqcup^\infty, n) \quad \text{if } \delta(p, y_n) = (q, b, N)$$

b) Turing Machines whose tape is infinitely long in *both* directions (not only to the right) and without a start marker $\vdash$, i.e. the initial tape content for input $x \in \Sigma^*$ is $\sqcup^\infty x \sqcup^\infty$ with the read/write head at the first character of x (or a $\sqcup$ if x is empty).

Formally: the tape content is a function $\mathrm{y} : \mathbb{Z} \to \Gamma$ which is initially

$$y_i = \begin{cases} x_i & \text{if } 0 \leq i < |x| \\ \sqcup & \text{otherwise} \end{cases}$$

and the step function is

$$(p, y, n) \xrightarrow[M]{1} \begin{cases} (q, y_{n:=b}, n-1) & \text{if } \delta(p, y_n) = (q, b, \mathrm{L}) \\ (q, y_{n:=b}, n+1) & \text{if } \delta(p, y_n) = (q, b, \mathrm{R}) \end{cases}$$

$$\text{where } (y_{n:=b})_i = \begin{cases} b & \text{if } i = n \\ y_i & \text{otherwise} \end{cases}$$

Hint: Show that two notions of Turing Machines are equivalent by transforming a Turing Machine according to one notion into one of the other notion that has the same input–output behaviour. An informal description of this process is sufficient and you do not have to prove that your process really does preserve input–output behaviour.