Recall the following:

- The notation RS is the *composition* of R and S, i.e.

$$RS = \{(x, z) \mid \exists y.\ (x, y) \in R \land (y, z) \in S\}$$

- R^n is R composed with itself n times, i.e. $R^0 = \{(x, x) \mid x \in A\}$ (the identity relation), $R^1 = R$, $R^{n+1} = RR^n$

1) Consider an arbitrary relation R on set X.
   a) Prove that for a *finite* X, there are natural numbers $i, j \in \mathbb{N}$, $i < j$, such that, $R^i = R^j$.
   b) Find a relation R on a finite set X such that $R^n \neq R^{n+1}$ for all $n > 0$.
   c) Show that the point a) above does not need to hold when X is *infinite*. This means, find a relation R on an infinite set X such that for all $i \neq j$ we have $R^i \neq R^j$.

2) Let $G = (V, E)$ be a digraph and define a relation $\preceq$ on its vertices such that $u \preceq v$ iff there exists a path from u to v. This relation is clearly reflexive and transitive (a so-called *preorder*).
   a) What are the minimal and maximal elements of $\preceq$?
   b) What conditions does G have to fulfil in order for $\preceq$ to be a partial order? Prove your answer!

   Let A be a finite set. We now define a relation $\preceq \subseteq 2^A \times 2^A$ such that $X \preceq Y$ iff there exists an injective function $f : X \to Y$. We also define $X \sim Y$ iff $X \preceq Y$ and $Y \preceq X$.

   c) Prove that $\preceq$ is a preorder, but not a partial order.
   d) What are the minimal and maximal elements of $\preceq$?
   e) When is $X \sim Y$?

3) For a Turing machine M, let $\mathcal{L}(M)$ be the set of words that M accepts. We say that a set of words $A \subseteq \Sigma^*$ is decidable if there exists a total Turing machine M such that $\mathcal{L}(M) = A$. In the following, you may assume that all Turing machines in this exercise operate on the same alphabet $\Sigma = \{0, 1\}$ and the same tape symbols $\Gamma = \{0, 1, \sqcup, \vdash\}$ for simplicity.
   a) Show that $\emptyset$ and Σ^* are decidable by drawing corresponding Turing machines.
   b) Show that if A is decidable, then so is its complement $\Sigma^* \setminus A$.
   c) Show that any finite set A is decidable by giving a corresponding Turing machine.
   d) Given two Turing machines M_1 and M_2 whose functions are $f : A \to B$ and $g : B \to C$, construct a Turing machine whose function is the composition of f and g, i.e. $f \circ g$. (Reminder: $(f \circ g)(x) = f(g(x))$)
   e) For a word $w = w_1 \dots w_n$, let $w^R = w_n \dots w_1$ be the reverse word, e.g. $(uibk)^R = kbiu$. Similarly, for a set of words A, let $A^R = \{w^R \mid w \in A\}$. Show that if A is decidable, then A^R is decidable.

   In all cases, if you have to give a Turing machine, an informal sketch is sufficient.