1) Consider a Turing machine M that cannot write onto the tape and can only move its head to the right. So the transition function of M would be $\delta(p, a) = (q, a, R)$. Moreover, we allow more than one accepting/rejecting state. If the Turing machine M is in an accepting/rejecting state, it can also leave this state. Every state that is not an accepting state is a rejecting state. With this specification, we can reduce some unnecessary information and define M as quintuple $M = (Q, \Sigma, \delta, s, F)$ where $F \subseteq Q$ is the set of accepting states.
Note that a Turing machine with such a specification is also called a Deterministic Finite Automata (DFA).

   a) Design a DFA for the set of strings in $\{0, 1, 2\}^*$ that accepts all strings that end with 01.
   b) Test your DFA on the input strings 201001 and 01210.
   c) Consider the language $L = \{0^n 1^n \mid n \geq 0\}$. Can you find a DFA? Are there other examples of such languages?

2) Recall that we defined R^+ as the smallest transitive relation that contains R. Prove the following alternative, more direct characterisation of R^*:

$$R^+ = \bigcup_{n \geq 1} R^n$$

To make things easier for you, we will guide you through the proof. We will denote the right-hand side of the above equation as RHS.

   a) Prove that RHS contains R. (this is very easy, don't get confused by that)
   b) Prove that RHS is transitive.
   c) Prove RHS is the smallest transitive relation containing R, i.e. that any transitive relation S that contains R is a superset of RHS.

You may use the following without proof:

- $y \in \bigcup_{x \in A} B_x \iff \exists x {\in} A.\ y \in B_x$
- $R^{m+n} = R^m R^n$ for any relation R and any $m, n \in \mathbb{N}$. (To get some intuition for this, think about what this means in a graph!)
- Relation powers are monotonic, i.e. $R \subseteq S \implies R^m \subseteq S^m$ for any relations R, S and any $m \in \mathbb{N}$.

3) The following is the bonus exercise from sheet 2, which we pose here again because the vast majority of students did not do it (and those who did made quite a few mistakes!). So here you have another chance to do it, and you are *strongly* encouraged to do so. We stress again that the question is whether these properties hold for **all** relations S and R, not just the ones from exercise 3 on sheet 2.

Hint: If you struggle with what these mean or coming up with counterexamples, try thinking of graphs instead of relations.

Which of the following statements are true for **all** relations R and S? Give a counterexample for the false ones! For the true ones, give an informal explanation for why they hold.

a) $(R^s)^* = (R^*)^s$

b) $(R^=)^n = (R^n)^=$

c) $(R \cup S)^* = (R^* S^*)^*$

d) $(R \cup S)^* = R^* \cup S^*$

e) $(R \cap S)^* = R^* \cap S^*$

f) $(RS)^* = (SR)^*$

Hint: When looking for counterexamples, consider the relations R and S from ex. 3 from sheet 2, the relation $T := \{(x, -x) \mid x \in \mathbb{Z}\}$, or try to come up with a suitable small graph.

4*) An *equivalence* on a set X is a relation on X which is reflexive, symmetric, and transitive. Let R and S be an *arbitrary* equivalences on a set X. Which of the following relations are also necessarily equivalences?

a) $R \cup S$

b) $R \cap S$

c) $R \setminus S$ (the elements of R without the elements of S)

d) RS