1) Show by well-founded induction that every natural number greater than 1 can be divided by a prime number.

2) Are the following relations well-founded relations, well-founded partial orders, or neither? Explain why!

   a) The divisibility relation on natural numbers.

   b) The "$<$" relation on the interval $[0, 1]$ of real numbers (i.e. $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$)

   c) $R_1 = \{((a, b), (c, d)) \mid a, b, c, d \in \mathbb{N},\ a < c\}$

   d) $R_2 = \{((a, b), (c, d)) \mid a, b, c, d \in \mathbb{N},\ a < c\ \wedge\ b < d\}$

   e) $R_3 = \{((a, b), (c, d)) \mid a, b, c, d \in \mathbb{N},\ a < c\ \vee\ b < d\}$

3) Let Σ be a finite set of symbols. Let us define the *maximo-lexicographical ordering* $<_{\text{mlex}}$ on the set of all finite words $\Sigma^\star$ as follows:

$$x <_{\text{mlex}} y \quad \iff \quad \ell(x) < \ell(y)\ \vee\ (\ell(x) = \ell(y)\ \wedge\ x <_{\text{lex}} y)$$

   where $<_{\text{lex}}$ is the lexicographical ordering from the lecture. Prove that $<_{\text{mlex}}$ is a well-founded total strict order.

4*) In the following, let R and S be relations over some set A. Which of these statements are true? Give an informal explanation for true statements and a counterexample for false ones.

   a) If R is well-founded, then every subset $R' \subseteq R$ is well-founded.

   b) If R and R' are well-founded, then $R \cup R'$ is well-founded.

   c) If R and S are well-founded, RS is well-founded.

   d) If R is well-founded, then for any function $f : B \to A$, the relation $R' = \{(x, y) \mid (f(x), f(y)) \in R\}$ is also well-founded.

   e) For any $n > 0$, R^n is well-founded if and only if R is well-founded.

   f) If R is well-founded, then R is acyclic (i.e. there is no x with $(x, x) \in R^+$).

   g) If A is finite and acyclic, then R is well-founded.