1) Let $G$ be the undirected weighted multigraph having vertices $V = \{0, 1, \ldots, 7\}$ and edges $E = \{0, 1, \ldots, 8\}$ with end-points $r$ and weights $b$ given by:

| $e$ | $r(e)$ | $b(e)$ |
|---|---|---|
| 0 | {0, 1} | 3 |
| 1 | {0, 2} | 2 |
| 2 | {0, 3} | 1 |
| 3 | {1, 3} | 2 |
| 4 | {2, 3} | 1 |
| 5 | {4, 5} | 2 |
| 6 | {4, 6} | 3 |
| 7 | {5, 7} | 1 |
| 8 | {6, 7} | 3 |

Compute a minimal spanning forest of $G$ using Kruskal's algorithm. Is it unique? How many possible spanning forests (not necessarily minimal) are there?

2) Consider a set $S = \{1, 2, \ldots, 16\}$ where we pick nine different numbers. Prove that there are always two picked numbers whose sum is 17.

Can you find a straightforward generalisation of your proof when $S = \{1, \ldots, 2n\}$?

3) We roll 5 dice after another, each having the numbers 1 to 6 on its faces. The dice are distinguishable (i.e. there is a first, second, third, etc. die) and we look at the results that they show and denote tham as e.g. 21135 (the first die shows a 2, the second one a 1, etc.) Note that e.g. the outcomes 12345 and 21345 are considered different.

There are $6^5 = 7776$ possible outcomes. Find out how many of these satisfy the following:

a) Yahtzee: all five dice show the same number

b) Large straight: The five dice, when ordered in a certain way, show consecutive numbers, e.g. 42153 or 23654

c) Four of a kind: At least four dice show the same number

d) Full House: Three dice show one number $a$ and the remaining two show another number $b$. Note that this means that a Yahtzee is *not* a full house.

e) Three of a kind: At least three dice show the same number

f) **Optional exercise (no points):** Small straight: We can pick four dice from our five and arrange them in a way so that they show consecutive numbers, e.g. 42613 or 54352. Note that this means that a large straight is also a small straight.
**Caution:** This one is a bit more difficult.

4*) Consider Kruskal's algorithm from the lecture.

a) Propose a modification of the algorithm to compute a spanning forest with a maximum possible weight instead of the minimal. Prove its correctness.

b) Prove the following. If the weight function is injective, then the minimal spanning forest is unique (that is, all other spanning forests are not minimal).

c) Is the following true? If the minimal spanning forest of a weighted multigraph is unique, then the weight function must be injective.

d) Consider the fully connected undirected graph $G_5$ on 5 nodes where every node is connected to every other node except itself. How many edges does $G_5$ have? How many spanning trees of $G_5$ are there?