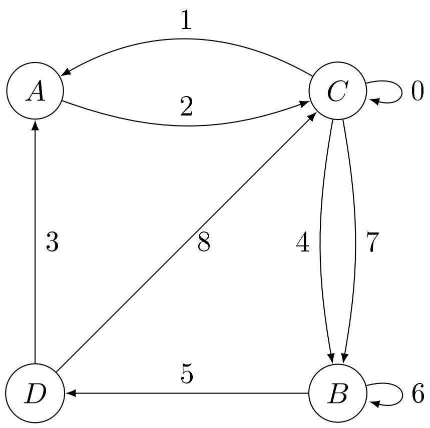
1) a) Visualisation:



b) Immediate predecessors: Node C and D
Immediate Successors: Node C

c) Indegree: 3
Outdegree: 2

d) Yes.

e) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

2) a)

$$A_0 = \begin{pmatrix} 0 & 2 & 5 & \infty & \infty \\ \infty & 0 & 1 & \infty & \infty \\ \infty & \infty & 0 & 2 & \infty \\ \infty & \infty & \infty & 0 & 3 \\ \infty & \infty & 2 & \infty & 0 \end{pmatrix} \qquad A_1 = A_0 \qquad A_2 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty \\ \infty & 0 & 1 & \infty & \infty \\ \infty & \infty & 0 & 2 & \infty \\ \infty & \infty & \infty & 0 & 3 \\ \infty & \infty & 2 & \infty & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 2 & 3 & 5 & \infty \\ \infty & 0 & 1 & 3 & \infty \\ \infty & \infty & 0 & 2 & \infty \\ \infty & \infty & \infty & 0 & 3 \\ \infty & \infty & 2 & 4 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 2 & 3 & 5 & 8 \\ \infty & 0 & 1 & 3 & 6 \\ \infty & \infty & 0 & 2 & 5 \\ \infty & \infty & \infty & 0 & 3 \\ \infty & \infty & 2 & 4 & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0 & 2 & 3 & 5 & 8 \\ \infty & 0 & 1 & 3 & 6 \\ \infty & \infty & 0 & 2 & 5 \\ \infty & \infty & 5 & 0 & 3 \\ \infty & \infty & 2 & 4 & 0 \end{pmatrix}$$

b)

$$A_0 = \begin{pmatrix} 0 & \infty & 2 & \infty & \infty \\ 3 & 0 & \infty & \infty & \infty \\ \infty & 2 & 0 & \infty & \infty \\ \infty & \infty & 1 & 0 & \infty \\ \infty & \infty & 5 & 2 & 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & \infty & 2 & \infty & \infty \\ 3 & 0 & 5 & \infty & \infty \\ \infty & 2 & 0 & \infty & \infty \\ \infty & \infty & 1 & 0 & \infty \\ \infty & \infty & 5 & 2 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & \infty & 2 & \infty & \infty \\ 3 & 0 & 5 & \infty & \infty \\ 5 & 2 & 0 & \infty & \infty \\ \infty & \infty & 1 & 0 & \infty \\ \infty & \infty & 5 & 2 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 4 & 2 & \infty & \infty \\ 3 & 0 & 5 & \infty & \infty \\ 5 & 2 & 0 & \infty & \infty \\ 6 & 3 & 1 & 0 & \infty \\ 10 & 7 & 5 & 2 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 4 & 2 & \infty & \infty \\ 3 & 0 & 5 & \infty & \infty \\ 5 & 2 & 0 & \infty & \infty \\ 6 & 3 & 1 & 0 & \infty \\ 8 & 5 & 3 & 2 & 0 \end{pmatrix} \quad A_5 = A_4$$

c) The matrices differ. The final matrices (A_5) contain the same numbers but in a different order. The intermediate matrices may contain different numbers. The meaning of the element $(A_k)_{i,j}$ is the length of the shortest path from node v_i to node v_j which uses only vertices $\{v_0, \ldots, v_{k-1}\}$ as the inner nodes (given the vertices are denoted $v_0, \ldots, v_4$).

3) **Note:** The following solution is much more formal and detailed than what is expected if you. However, since learning how to write rigorous proofs is one of the goals of the lecture, we show you this so that you learn the approach and the vocabulary involved in such proofs.

a) Suppose we had a cycle $(v_1, v_2, \ldots, v_{n-1}, v_1)$. Since G is an arborescence, there exists exactly one path from v_0 to v_1. However, if we add our cycle to the end of this path, we also get a path from v_0 to v_1. Since paths are, by definition, non-empty, this really is a different path, and we obtain a contradiction.

b) For $1 \leq i \leq k$ let G_i be the graph consisting of all vertices reachable from v_i and the edges between them.

The subgraphs really are subgraphs:
What we need to show: Every vertex and edge in G_i is part of G'. (i.e. the vertex v_0 we deleted does not occur in any vertex or edge)
Proof:

- $v_0 \notin G_i$ since otherwise we would have a cycle going from v_0 to v_i and back.
- There is no edge of the form $v \to v_0$ or $v_0 \to v$ in G_i since that would require v_0 to be reachable from v_i, giving us a cycle.

Disjointness of vertices:
What we need to show: No vertex is in two of the G_i.
Proof: Suppose a vertex v were both in G_i and G_j with $i \neq j$. Then there is one path from v_i to v in G_i and one from v_j to v in G_j, leading two two different paths from v_0 to v in G.

The subgraphs span G:
What we need to show: Every vertex and edge of G' is in one of the G_i.
Proof:

- Let v be a vertex of G'. There is a path from v_0 to v in G. That path is non-empty and its first edge must be of the shape (v_0, v_i) for $1 \leq i \leq k$ and thus v is reachable from v_i.
- Let $v \to v'$ be an edge of G'. As we have just shown, there is a path $v_i \to \ldots \to v$ in some G_i and thus v' is reachable from v_i and thus the edge (v, v') is also in G_i.

The subgraphs are arborescences:
What we need to show: There is exactly one path from v_i to every vertex v of G_i.
Proof: By definition of G_i, there exists at least one path from v_i to v. Thus, we only need to show that this path is unique. Consider therefore two paths p_1 and p_2 from v_i to v in G_i. By adding the edge (v_0, v_i) at the beginning, we obtain two paths p'_1 and p'_2 from v_0 to v in G. Since G is an arborescence, we must have $p'_1 = p'_2$ and therefore (by removing the first edge again) also $p_1 = p_2$.

4∗) a) Yes, the algorithm still produces the correct result.

b) No, the algorithm does not produce the correct result. For example, the shortest path from `d` to `e` is not computed correctly, because the cycle `d - e - c - d` can be iterated arbitrary times, resulting still in shorter and shorter paths.

c) Negative weights are acceptable as long as there is no negative cycle, that is, a cycle with a negative sum of weights. This is because the Floyd–Warshall algorithm only considers paths on which every vertex occurs at most once (i.e. it assumes that a shortest path cannot contain the same vertex twice). Negative cycles can be detected by the algorithm by checking for negative numbers on the diagonal after the algorithm has finished.