1) Show that for any infinite set A and any $x \in A$, we have $|A \setminus \{x\}| = |A|$.
Hint: Note that any infinite set has a countably infinite subset. Try proving that the statement holds for all countably infinite sets A first.

Solution:
First of all, let us show that the statement shows for every countably infinite set A. Then $A \setminus \{x\}$ is

- countable, since it is a subset of A, which is countable by assumption
- infinite, since if $A \setminus \{x\}$ were finite, then $A \setminus \{x\} \cup \{x\} = A$ would be as well.

Since all countably infinite sets have the same cardinality (namely $|\mathbb{N}|$), we are done.

Now for the proof of the more general case where A is infinite, but not necessarily countable: Following the hint, let X be a countably infinite subset of X. We also assume w.l.o.g. that $x \in X$. Such a set can be constructed by just starting with $x_0 := x$ and then iteratively picking some $x_{i+1} \in A \setminus \{x_0, \dots, x_i\}$.

Now, let $Y := A \setminus X$. As we have just shown, $|X| = |X \setminus \{x\}|$, so there exists a bijection $f : X \to X \setminus \{x\}$. Then the function

$$g(y) = \begin{cases} f(y) & \text{if } y \in X \\ y & \text{if } y \in Y \end{cases}$$

is a bijection between $X \cup Y$ and $X \setminus \{x\} \cup Y$, i.e. between A and $A \setminus \{x\}$.

2) a) Show that the set of rational numbers $\mathbb{Q}$ is countably infinite.
b) Let $\mathcal{G}$ be the set of graphs $G = (V, E)$ with $V \subseteq \mathbb{N}$. Show that $\mathcal{G}$ is countably infinite.
Remark: When we posed this exercise, we forgot to say that we meant *finite* graphs. If the graphs can be infinite, this does not hold.
c) Show that $|[0, 1]| = |(0, 1)|$
d) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Show that $|[a, b]| = |[c, d]|$.
e) Show that $|[0, 1]| = |\mathbb{R}|$.

Hint: Recall that there are two basic ways of showing that two sets have the same cardinality:

- directly, i.e. by giving a bijection between them, or
- through the Schröder–Bernstein theorem, i.e. showing $|A| \leq |B|$ and $|B| \leq |A|$ by giving injective functions $f : A \to B$ and $g : B \to A$

Recall that $[a, b] = \{x \mid x \in \mathbb{R}, a \leq x \leq b\}$ and $(a, b) = \{x \mid x \in \mathbb{R}, a < x < b\}$.

Solution:

a) It is obvious that $\mathbb{Q}$ is infinite (after all, the natural numbers are a subset of it). It remains to show that it is countable. Clearly, the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable due to what we learnt in the lectures. Consider the function

$$f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}, f(a, b) = \frac{a}{b}\ .$$

This function is surjective, i.e. $\mathbb{Q}$ is the image of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ under f, and we have seen in the lecture that the image of a countable set is countable.

Remark: It is also possible to construct an explicit bijection between the naturals and the rationals, but that is more work (cf. Stern–Brocot Tree).

b) It is clear that $\mathcal{G}$ is infinite, as evidenced e.g. by the sequence of graphs $(G_i)_{i\in\mathbb{N}}$ where each G_i consists of a single node i and no edges. It remains to show that the set is countable.

For any natural number n, let $\mathcal{G}_n$ denote the set of graphs whose vertices are a subset of $I_n := \{0, \dots, n\}$. Clearly, for every graph $G \in \mathcal{G}$, there exists an n such that $G \in \mathcal{G}_n$. Thus $\mathcal{G} = \bigcup_{n\geq 0} \mathcal{G}_n$. Moreover, $\mathcal{G}_n$ is finite for any n since $\mathcal{G}_n \subseteq 2^{I_n} \times 2^{I_n \times I_n}$. From the lecture, we have seen that the union of countably many countable sets is again countable.

Caution: As was mentioned before, this only holds if the graphs are assumed to be finite. If the graphs can be infinite, the statement is in fact wrong: the function

$$f : \mathcal{P}(\mathbb{N}) \to \mathcal{G},\ f(X) = (X, \emptyset)$$

is then an injective map from $\mathcal{P}(\mathbb{N})$ to $\mathcal{G}$, and since $\mathcal{P}(\mathbb{N})$ is uncountable, so is $\mathcal{G}$.

c) Follows directly from Exercise 1). Alternatively: $|(0, 1)| \leq |[0, 1]|$ because the left-hand side is a subset of the right-hand side, and also $|[0, 1]| = |(\frac{1}{3}, \frac{2}{3})|$ using Exercise 2d) and $|(\frac{1}{3}, \frac{2}{3})| \leq |(0, 1)|$ because subset. Then $|(0, 1)| = |[0, 1]|$ follows from Schröder–Bernstein.

d) For any $a < b$, the function $f : [a, b] \to [0, 1],\ f(x) = \frac{x-a}{b-a}$ is a bijection. Thus, $|[a, b]| = |[0, 1]| = |[c, d]|$.

e) From the previous exercises, we know that $|[0, 1]| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$, and the tangent function is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. Another possible bijection would be $f : (-1, 1) \to \mathbb{R},\ x \mapsto \frac{x}{1-|x|}$.

3) State if the following statements are true or false. Justify your answer with a proof:

a) $\mathbb{R}$ is countable.

b) $\mathbb{N} \times \mathbb{Q}$ is countable.

c) If A is countable but B is not, then $A \cap B$ is countable.

d) If A is a infinite set, then A^* is countable.
Remark: Here, A^* denotes the set of all finite lists with elements from A. A finite list of length n can be thought of as a function $\{1, \dots, n\} \to A$, or as an n-tuple with elements of A.

Solution:

a) False. Proof by Cantor diagonalisation.

b) True. Since the cartesian product of finitely many countable sets, is countable. And we know $\mathbb{N}$ is countable and $\mathbb{Q}$ is countable from Exercise 2a.

c) True. Since every subset of a countable set is countable. We know $(A \cap B) \subseteq A$.

d) False. Assume A is not countable (e.g. $\mathbb{R}$) but A^* is. Since every subset of a countable set is countable and $A \subseteq A^*$, also A is countable, which is a contradiction to our assumption.

4*) Suppose that in an infinitely near future, the Discrete Structure course becomes so popular that countably infinitely many students want to attend it. A new lecture room is built, with countably infinitely many seats numbered by natural numbers. At the first lecture, all the seats are taken but one student arrives late and has no place to sit. The lecturer asks everyone to stand up and move from seat n to seat $n+1$. This makes the seat number 0 unoccupied and the late student can sit there. Now, you – the best student from this year – are asked to take care of other late students, and find some place for them to sit. How would you handle the following situations?

a) 10 more students arrive late. How will you move the present students, and where will the i-th late student sit?

b) A bus with countably infinitely many students arrives late. How will you move the students, and where will the i-th student from the bus sit?

c) Countably infinitely many buses, each with countably infinitely many students, arrive late. How will you move the present students, and where will you place student i from bus j? Don't worry if some places are left unoccupied afterwards.

Solution:

a) A student from seat n will move to the seat $n + 10$. The i-th late student will sit on an empty seat i.

b) A student from seat n will move to the seat $2n$. Now all the odd seats are empty. The i-th late student will sit on an empty seat $2i + 1$.

c) A student from seat n will move to the seat $2n$. Again, all the odd seats are empty now. The i-th student from bus 0 will sit on the seat 3^{i+1} (we are counting from 0). For the next bus (that is, bus number 1) we will take the next prime number, that is 5, and the i-th student from bus 1 will sit on the seat 5^{i+1}. In general, student i from bus j will sit on the seat p_j^{i+1} where $p_0 = 3$ and p_{n+1} is the smallest prime number greater than p_n.