Recall the following:

- The notation RS is the *composition* of R and S, i.e.

$$RS = \{(x, z) \mid \exists y.\ (x, y) \in R \land (y, z) \in S\}$$

- R^n is R composed with itself n times, i.e. $R^0 = \{(x, x) \mid x \in A\}$ (the identity relation), $R^1 = R$, $R^{n+1} = RR^n$

1) Consider an arbitrary relation R on set X.

   a) Prove that for a *finite* X, there are natural numbers $i, j \in \mathbb{N}$, $i < j$, such that, $R^i = R^j$.

   b) Find a relation R on a finite set X such that $R^n \neq R^{n+1}$ for all $n > 0$.

   c) Show that the point a) above does not need to hold when X is *infinite*. This means, find a relation R on an infinite set X such that for all $i \neq j$ we have $R^i \neq R^j$.

   Solution:

   a) Clearly there are only finitely many relations on a finite set X, namely $m = 2^{(|X|^2)}$. Hence among the first relations R^1,R^2,...,$R^{(m+1)}$ at least two must be equal.

   b) Consider a "triangle" relation $R = \{(a, b), (b, c), (c, a)\}$ on the set $X = \{a, b, c\}$.

   c) Consider the relation $R = \{(m, m + 1) \mid m \in \mathbb{N}\}$ on natural numbers. Then $R^n = \{(m, m + n) \mid m \in \mathbb{N}\}$ can be easily proved by induction on n. Hence the claim.

2) **Caution:** Sub-exercise a) was modified slightly.
   Let $G = (V, E)$ be a digraph and define a relation $\preceq$ on its vertices such that $u \preceq v$ iff there exists a path from u to v. This relation is clearly reflexive and transitive (a so-called *preorder*).

   a) What are the minimal and maximal elements of $\preceq$ if G is cycle-free?

   b) What conditions does G have to fulfil in order for $\preceq$ to be a partial order? Prove your answer!

   Let A be a finite set. We now define a relation $\preceq \subseteq 2^A \times 2^A$ such that $X \preceq Y$ iff there exists an injective function $f : X \to Y$. We also define $X \sim Y$ iff $X \preceq Y$ and $Y \preceq X$.

   c) Prove that $\preceq$ is a preorder, but not a partial order.

   d) What are the minimal and maximal elements of $\preceq$?

   e) When is $X \sim Y$?

   Solution:

   a) The minimal elements are the vertices with indegree 0, the maximal ones those with outdegree 0. Of course, it is possible that no such vertices exist. In a graph that contains cycles, the situation is more complicated.

   b) $\preceq$ is a partial order if (and only if) the G contains no cycle (other than possibly loops, i.e. transitions of the form $v \to v$).
   Direction 1: preorder $\Longrightarrow$ cycle-free
   Suppose G had a cycle that is not a loop. Then $u \preceq v$ and $v \preceq u$ for all vertices on the

cycle, which contradicts antisymmetry.
Direction 2: cycle-free $\Longrightarrow$ preorder
Reflexivity and transitivity are already given, so we need only show antisymmetry, i.e.

$$\forall u, v{\in}V.\ u \preceq v \wedge v \preceq u \Longrightarrow u = v$$

or less formally:

$$\text{if } u \preceq v \text{ and } v \preceq u \text{ for any two vertices } u, v \text{ then } u = v$$

To prove that this is the case, let u, v be arbitrary vertices with $u \preceq v$ and $v \preceq u$. Thus we have a path from u to v and from v back to u. If $u \neq v$, we would have a cycle that is not a loop, but we have cycle-freeness as an assumption. Hence $u = v$.

c) **Reflexivity:** The identity function $x \mapsto x$ is an injection from any set X to itself.

Transitivity: Given injections $f : X \to Y$ and $g : Y \to Z$, their composition $g \circ f$ is an injection from X to Z since $g(f(x)) = g(f(y)) \Longleftrightarrow f(x) = f(y) \Longleftrightarrow x = y$.

Antisymmetry: Let $A = \{0, 1\}$. Then obviously $0 \mapsto 1$ is an injection from $\{0\}$ to $\{1\}$ and $1 \mapsto 0$ is an injection from $\{1\}$ to $\{0\}$, but $\{0\} \neq \{1\}$.

d) The empty set is a minimal element, since the empty function is an injection from the empty set into any other set. This is the only minimal element, since there exists no function $X \to \emptyset$ (let alone an injection). The full set A is maximal, since the identity function is an injection from any subset $X \subseteq A$ into A. This is the only maximal element, since if $f : X \to Y$ is an injection, $|X| \leq |Y|$ and clearly any proper subset of A has less elements than A.

Note: This all becomes trickier if we consider infinite sets. For instance, if $A = \mathbb{N}$, the sets $\mathbb{N} \setminus \{0\}$ and $\{n \in \mathbb{N} \mid n \text{ prime}\}$ are also maximal. But more about this later in the lecture!

e) An injection from X to Y exists iff Y has at least as many elements as X; to see this, one can simply arrange all the elements of X and Y as lists and then start associating them with one another. This then means that $X \sim Y$ iff $|X| = |Y|$.

3) For a Turing machine M, let $\mathcal{L}(M)$ be the set of words that M accepts. We say that a set of words $A \subseteq \Sigma^*$ is decidable if there exists a total Turing machine M such that $\mathcal{L}(M) = A$. In the following, you may assume that all Turing machines in this exercise operate on the same alphabet $\Sigma = \{0, 1\}$ and the same tape symbols $\Gamma = \{0, 1, \sqcup, \vdash\}$ for simplicity.

   a) Show that $\emptyset$ and Σ^* are decidable by drawing corresponding Turing machines.
   b) Show that if A is decidable, then so is its complement $\Sigma^* \setminus A$.
   c) Show that any finite set A is decidable by giving a corresponding Turing machine.
   d) Given two Turing machines M_1 and M_2 whose functions are $f : A \to B$ and $g : B \to C$, construct a Turing machine whose function is the composition of f and g, i.e. $f \circ g$. (Reminder: $(f \circ g)(x) = f(g(x))$)
   e) For a word $w = w_1 \dots w_n$, let $w^R = w_n \dots w_1$ be the reverse word, e.g. $(uibk)^R = kbiu$. Similarly, for a set of words A, let $A^R = \{w^R \mid w \in A\}$. Show that if A is decidable, then A^R is decidable.

In all cases, if you have to give a Turing machine, an informal sketch is sufficient.

Solution:

a) Let the initial state be the rejecting (resp. the accepting) state.

b) Let M be a Turing machine with $\mathcal{L}(M) = A$. Swap the accepting and the rejecting state and we have a machine that decides $\Sigma^* \setminus A$.

c) Let n be the length of the longest word. Create a state q_u for every word $u \in \Sigma^*$ with $|u| \leq n$ and transitions $q_u \xrightarrow{0/0/R} q_{u0}$ and $q_u \xrightarrow{1/1/R} q_{u1}$ and $q_u \xrightarrow{\sqcup/\sqcup/R} t$ if $u \in A$ and $q_u \xrightarrow{\sqcup/\sqcup/R} r$ otherwise. The initial state is q_ε. This machine is clearly total since we end up in the accepting or rejecting state after at most $n+1$ steps. See Figure **??** for an example.

d) We make our lives a bit easier by allowing N-transitions that do not move the tape head (cf. exercise sheet 2 that one can get rid of these). Suppose $M_1 = (Q_1, \Sigma, \Gamma, \vdash, \sqcup, \delta_1, s_1, t_1, r_1)$ and $M_2 = (Q_2, \Sigma, \Gamma, \vdash, \sqcup, \delta_2, s_2, t_2, r_2)$. We assume w.l.o.g. that Q_1 and Q_2 are disjoint (otherwise we simply rename the states in Q_2 until this is the case). We now define

$$M := (Q_1 \cup Q_2, \Sigma, \Gamma, \vdash, \sqcup, \delta, s_2, t_1, r_2)$$

where

$$\delta(q, x) = \begin{cases} \delta_1(q, x) & \text{if } q \in Q_1 \\ \delta_2(q, x) & \text{if } q \in Q_2 \setminus \{t_2, r_2\} \\ (r_1, x, N) & \text{if } q = r_2 \\ (s_1, x, N) & \text{if } q = t_2 \end{cases}$$

In other words: We simply

- draw the two Turing machines next to each other,
- start in the initial state of M_2,
- connect the rejecting state of M_2 with the rejecting state of M_1,
- connect the accepting state of M_2 with the initial state of M_1,
- and make the rejecting and accepting states of M_2 the rejecting and accepting states of the new machine.

e) Let M be a Turing machine with $\mathcal{L}(M) = A$. We can construct a machine M' that computes the function $f : \Sigma^* \to \Sigma^*, w \mapsto w^R$ by

- marking all the 0s and 1s on the tape in some way, e.g. replacing 0 with $\hat{0}$ and 1 with $\hat{1}$
- swapping the first and last marked letter on the tape and remove their markings
- stop when there are no more marked letters and go back to the left

We then need only compose M' and M, as discussed in exercise d).

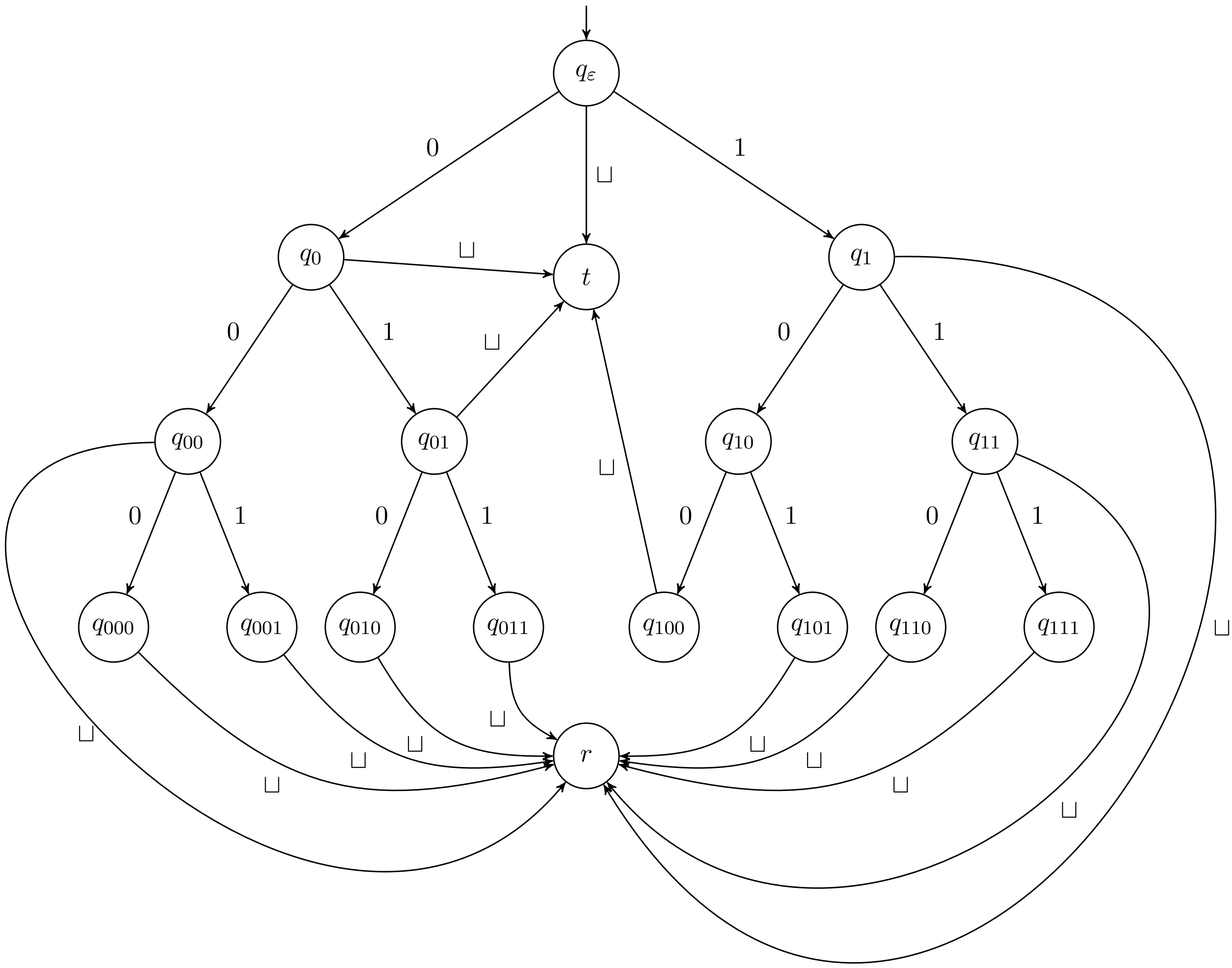

Figure 1: A Turing Machine that decides the set $\{\varepsilon, 0, 01, 100\}$