1) Consider a Turing machine M that cannot write onto the tape and can only move its head to the right. So the transition function of M would be $\delta(p, a) = (q, a, R)$. Moreover, we allow more than one accepting/rejecting state. If the Turing machine M is in an accepting/rejecting state, it can also leave this state. Every state that is not an accepting state is a rejecting state. With this specification, we can reduce some unnecessary information and define M as quintuple $M = (Q, \Sigma, \delta, s, F)$ where $F \subseteq Q$ is the set of accepting states.
Note that a Turing machine with such a specification is also called a Deterministic Finite Automata (DFA).

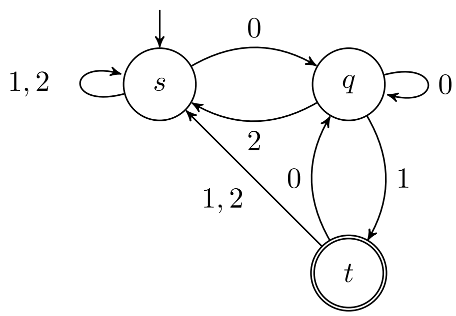
   a) Design a DFA for the set of strings in $\{0, 1, 2\}^*$ that accepts all strings that end with 01.
   b) Test your DFA on the input strings 201001 and 01210.
   c) Consider the language $L = \{0^n 1^n \mid n \geq 0\}$. Can you find a DFA? Are there other examples of such languages?

Solution:

   a) $M = \{Q, \Sigma, \delta, s, F\}$ where $Q = \{s, q, t\}$, $\Sigma = \{0, 1, 2\}$, $F = \{t\}$ and δ is given by the transition table:

δ	0	1	2
s	q	s	s
q	q	t	s
t	q	s	s

The following shows a graphical representation of the DFA:



   b) $s \ {}^{2}\ s \ {}^{0}\ q \ {}^{1}\ t \ {}^{0}\ q \ {}^{0}\ q \ {}^{1}\ t \in L(M)$ $\qquad$ $s \ {}^{0}\ q \ {}^{1}\ t \ {}^{2}\ s \ {}^{1}\ s \ {}^{0}\ q \notin L(M)$
   c) No, we cannot find a DFA. If one existed, lets assume such a TM with n states that only goes to the right exists. Then, lets consider the word prefixes $0^1..0^{n+1}$. Since there are more prefixes than states, there is at least one repeated. Lets say that after processing 0^k or 0^l the machine is in state x. Now since $0^k 1^k$ is accepting this means that $0^l 1^k$ would continue from the same state and would also be accepting. This is a contradiction, so such a DFA cannot exist.

2) Recall that we defined R^+ as the smallest transitive relation that contains R. Prove the following alternative, more direct characterisation of R^*:

$$R^+ = \bigcup_{n \geq 1} R^n$$

To make things easier for you, we will guide you through the proof. We will denote the right-hand side of the above equation as RHS.

a) Prove that RHS contains R. (this is very easy, don't get confused by that)

b) Prove that RHS is transitive.

c) Prove RHS is the smallest transitive relation containing R, i.e. that any transitive relation S that contains R is a superset of RHS.

You may use the following without proof:

- $y \in \bigcup_{x \in A} B_x \iff \exists x{\in}A.\ y \in B_x$
- $R^{m+n} = R^m R^n$ for any relation R and any $m, n \in \mathbb{N}$. (To get some intuition for this, think about what this means in a graph!)
- Relation powers are monotonic, i.e. $R \subseteq S \implies R^m \subseteq S^m$ for any relations R, S and any $m \in \mathbb{N}$.

Solution:

a) $R = R^1 \subseteq \text{RHS}$ *q.e.d.*

b) Let $(x, y) \in \bigcup_{n \geq 1} R^n$ and $(y, z) \in \bigcup_{n \geq 1} R^n$. Then there exist $n_1, n_2 \in \mathbb{N}_{\geq 1}$ such that $(x, y) \in R^{n_1}$ and $(y, z) \in R^{n_2}$. Therefore, $(x, z) \in R^{n_1} R^{n_2} = R^{n_1 + n_2} \subseteq \bigcup_{n \geq 1} R^n$. *q.e.d.*

c) We prove that S is a superset of RHS by showing that any element of RHS is also an element of S.

First of all, we note that $S^2 \subseteq S$ because S is transitive (this is, in fact, the definition of transitivity). By applying this $n - 1$ times, we have $S^n \subseteq S$ for any $n \geq 1$.

Now let $(x, y) \in \bigcup_{n \geq 1} R^n$. Then there exists an $m \in \mathbb{N}_{\geq 1}$ such that $(x, y) \in R^m \subseteq S^m \subseteq S$, which is what we had to show. *q.e.d.*

If you want a more formal proof that $S^n \subseteq S$, we can also show this by induction. The base cases are $n = 1$ and $n = 2$, i.e. $S^1 \subseteq S$ and $S^2 \subseteq S$, which are obvious. In the induction step, we have to show that $S^n \subseteq S$ for any $n \geq 2$ with the induction hypothesis $S^{n-1} \subseteq S$. We have $S^n = S^2 S^{n-2} \subseteq S S^{n-2} = S^{n-1} \subseteq S$.

3) The following is the bonus exercise from sheet 2, which we pose here again because the vast majority of students did not do it (and those who did made quite a few mistakes!). So here you have another chance to do it, and you are *strongly* encouraged to do so. We stress again that the question is whether these properties hold for **all** relations S and R, not just the ones from exercise 3 on sheet 2.

Hint: If you struggle with what these mean or coming up with counterexamples, try thinking of graphs instead of relations.

Which of the following statements are true for **all** relations R and S? Give a counterexample for the false ones! For the true ones, give an informal explanation for why they hold.

a) $(R^s)^* = (R^*)^s$

b) $(R^=)^n = (R^n)^=$

c) $(R \cup S)^* = (R^* S^*)^*$

d) $(R \cup S)^* = R^* \cup S^*$

e) $(R \cap S)^* = R^* \cap S^*$

f) $(RS)^* = (SR)^*$

Hint: When looking for counterexamples, consider the relations R and S from ex. 3 from sheet 2, the relation $T := \{(x, -x) \mid x \in \mathbb{Z}\}$, or try to come up with a suitable small graph.

Solution:

a) False, e.g. for $R = \{(0, 2), (1, 2)\}$.

b) False, e.g. $(0, 1) \in (R^=)^2$ but $(0, 1) \notin (R^2)^=$

c) True. On the left-hand side, we have every "path" consisting of steps from R and S. On the right-hand side, we have every path that can be split up into subpaths p_i that can again be split into a path in R and a path in S. It is clear that both are equivalent: every path on the right-hand side clearly has every step either in R or in S, and on the left-hand side we can simply split a path of length n into n subpaths of length 1.

For a formal proof we first note some auxiliary facts:

- The reflexive transitive closure is monotonic, i.e. $R \subseteq S \Longrightarrow R^* \subseteq S^*$. This can be seen e.g. by noting that for any relation R we have $R^* = \bigcup_{n \geq 0} R^n$ (cf. exercise 2) and the power operation on relations is monotonic.
- $R^*R^* = R^*$. The direction $R^*R^* \subseteq R^*$ holds because R^* is transitive, and the direction $R^* \subseteq R^*R^*$ holds because R^* is reflexive.
- $(R^*)^* = R^*$. This is because the reflexive transitive closure of a reflexive transitive relation is the relation itself (otherwise it would not be minimal).

Proof of $(R \cup S)^* = (R^*S^*)^*$:

Direction 1: $(R \cup S)^* \subseteq (R^*S^*)^*$
We have:

$$R \cup S = R^1S^0 \cup R^0S^1 \subseteq R^*S^* \cup R^*S^* = R^*S^*$$

and the rest follows by monotonicity of the reflexive transitive closure.

Direction 2: $(R^*S^*)^* \subseteq (R \cup S)^*$
We have:

$$(R^*S^*)^* \subseteq ((R \cup S)^*(R \cup S)^*)^* = ((R \cup S)^*)^* = (R \cup S)^*$$

d) False, e.g. with R and S from ex. 3 on sheet 2: $R^* \cup S^*$ is just two lines, but $(R \cup S)^*$ is the entire upper right quadrant. Or a shorter one: $R = \{(1, 2)\}$ and $S = \{(2, 3)\}$.

e) False. Counterexample: $R = \{(1, 3), (3, 2)\}$ and $S = \{(1, 2)\}$. Then $(1, 2) \in R^* \cap S^*$, but $(1, 2) \notin (R \cap S)^*$.

f) False. Counterexample: $R = \{(1, 0)\}$ and $S = \{(0, 0)\}$.

4*) An *equivalence* on a set X is a relation on X which is reflexive, symmetric, and transitive. Let R and S be an *arbitrary* equivalences on a set X. Which of the following relations are also necessarily equivalences?

a) $R \cup S$

b) $R \cap S$

c) $R \setminus S$ (the elements of R without the elements of S)

d) RS

Solution:

a) No. Let $X = \{a, b, c\}$. Let R be the transitive, symmetric, and reflexive closure of $\{(a, b)\}$ on X, and let S be the transitive, symmetric, and reflexive closure of $\{(b, c)\}$. We have $(a, b) \in R$ and $(b, c) \in S$ but no $(a, c) \in R \cup S$.

b) Yes. **Reflexivity:** Let $x \in X$. Clearly $(x, x) \in R$ and $(x, x) \in S$. Hence the claim. **Symmetry:** Let $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Thus $(y, x) \in R$ and $(y, x) \in S$ by the symmetry of R and S. Hence the claim. **Transitivity:** Let $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. By the transitivity of R and S we get both $(x, z) \in R$ and $(x, z) \in S$. Hence the claim.

c) No. Clearly not reflexive because all the elements $(x, x) \in S$ are removed.

d) No. Again take the relations from point (a). We have $(a, c) \in RS$ but $(c, a) \notin RS$.