Please note: Be very clear about what kind of induction you are performing and on what!

1) Prove by induction: For any natural numbers x and y, there exist integers a and b such that $ax + by = \gcd(x, y)$.

   You may use without proof the following facts for any $x, y \in \mathbb{N}$:

   - $\gcd(x, y) = \gcd(y, x)$
   - $\gcd(x, 0) = x$
   - $\gcd(x, y) = \gcd(x, y - x)$

   Solution:
   We prove the statement by well-founded induction over (x, y) with the following relation:

   $$(x', y') \prec (x, y) \iff x' + y' < x + y$$

   Induction step: let x, y be arbitrary natural numbers.
   What we need to show: $\exists a\, b \in \mathbb{Z}.\ ax + by = \gcd(x, y)$

   Induction hypothesis: $\forall x'\, y'.\ (x', y') \prec (x, y) \implies \exists a\, b \in \mathbb{Z}.\ ax + by = \gcd(x', y')$.

   Proof of induction step:
   Let us assume w.l.o.g. that $x \leq y$. We can do this because addition and gcd are both commutative (cf. the hint).

   If $x = 0$, we obviously have $0 \cdot x + 1 \cdot y = y = \gcd(x, y)$.
   If, $x > 0$, we can apply our induction hypothesis to the pair $(x, y - x)$ to obtain a and b such that $ax + b(y - x) = \gcd(x, y - x)$. Rearranging and using the hint that $\gcd(x, y) = \gcd(x, y - x)$ gives us:

   $$(a - b)x + by = ax + b(y - x) = \gcd(x, y - x) = \gcd(x, y)$$

   And with that, we are done. *q.e.d.*

   Remark: This theorem is also known as *Bézout's lemma.*

2) Find and prove a relationship between the number of connected components in an undirected forest and its number of nodes and edges?
   Hint: Look at the lemma proved in the lecture for trees and try some example forests.

   Solution:
   Let $G = (V, E)$ be an undirected finite forest and let n be the number of connected components of G (where $n = 0$ for the empty graph). Then we have $|V| = |E| + n$.

   Proof: Let $G_1, \ldots, G_n$ be the connected components of G with $G_i = (V_i, E_i)$. Every G_i is connected (by definition) and cycle-free (since G is already cycle-free, since it is a forest). From the properties of trees we proved in the lecture, we then know that $|V_i| = |E_i| + 1$. Summing this equation for all i, we obtain

   $$\sum_{i=1}^{n} |V_i| = \sum_{i=1}^{n} (|E_i| + 1) = \sum_{i=1}^{n} |E_i| + n$$

Since the G_i are pairwise disjoint and, taken together, give us all of G, the sum on the left-hand side is simply $|V|$ and the sum on the right-hand side is clearly $|E|$, and we are done. *q.e.d.*

Note: Of course, other kinds of proofs also work, e.g. an induction over the number of nodes where we remove a leaf in every step.

3) Show that a undirected finite graph G is a forest if and only if every non-empty subgraph of G contains a vertex of degree ≤ 1.

Solution:
Direction 1: G is forest $\Rightarrow$ every non-empty subgraph of G contains a vertex of degree ≤ 1
Proof by contradiction: Assume that G is a forest but there exists a non-empty subgraph H where all vertices have degree ≥ 2. Then we can construct a path where we start with any node in H and keep extending the path with a different edge until we end up at a node that we have already seen. Since there are only finitely many nodes, this will happen eventually, and at that point we have found a cycle. But this contradicts the cycle-freeness of G.

Direction 2: Every non-empty subgraph of G contains a vertex of degree $\leq 1 \Rightarrow G$ is forest
We must show that G is cycle-free. Thus, suppose G had a cycle. That cycle is a non-empty subgraph of G in which every vertex has degree 2. This contradicts our assumption.

Hence we showed that a graph G is a forest if and only if every subgraph of G contains a vertex of degree ≤ 1.

4*) Prove by induction: Let x, y be words over some alphabet Σ such that $xy = yx$. Then there exist a word z and natural numbers m, n such that $x = z^m$ and $y = z^n$.

Solution:
We prove the statement by well-founded induction over the pair (x, y) with the following relation:

$$(x', y') \prec (x, y) \Longleftrightarrow |x'| + |y'| < |x| + |y|$$

Induction step: Let $x, y \in \Sigma^*$ be arbitrary words for which $xy = yx$.
What we need to show: $\exists z\, m\, n.\ x = z^m \wedge y = z^n$

Induction hypothesis: $\forall x'\, y'.\ (x', y') \prec (x, y) \wedge x'y' = y'x' \Longrightarrow \exists z\, m\, n.\ x' = z^m \wedge y' = z^n$

Proof of induction step:
If $x = \epsilon$, we can set $z := y$, $m := 0$, $n := 1$ and we are done. Analogously, if $y = \epsilon$, we can set $z := y$, $m := 1$, $n := 0$ and we are done. It remains to handle the case that $x \neq \epsilon$ and $y \neq \epsilon$.

If $|x| \leq |y|$, we define $k := |x|$ and $l := |y|$. Because $xy = yx$, the first k letters of xy must be equal to the first k letters of yx, i.e. the first k letters of y must be exactly x. Thus, if we define $u := y_n \ldots y_{l-1}$ (the remaining letters of y), we have $y = xu$.

If we plug this equation into $xy = yx$, we get $xxu = xux$. We cancel the factor x and get $xu = ux$. Since we know that x is non-empty and $y = xu$, we know that $|u| < |y|$ and therefore $(x, u) \prec (x, y)$. By induction hypothesis, there exist z, m, n such that $x = z^m$ and $u = z^n$. But then $y = xu = z^m z^n = z^{m+n}$ and we are done.

The case $|x| > |y|$ is completely analogous. *q.e.d.*