Summary last week (1/2)

- equivalence relation if reflexive, transitive, and symmetric
- if $\sim$ equivalence on A, then $[a] = \{b \mid a \sim b\}$ is equivalence class of $a \in A$
- b representative of $[a]$ if $b \in [a]$
- B system of representatives picks for all $a \in A$, unique representative b of $[a]$ in B

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- B system of representatives picks for all $a \in A$, unique representative b of $[a]$ in B
- enumeration of set A is bijection from (initial segment of) $\mathbb{N}$ to A; A countable
- if initial segment, then A finite, otherwise countably infinite
- countability preserved by cartesian product (dove tailing)

Theorem

1. *Every subset of a countable set is countable.*
2. *The image of a countable set if countable.*
3. *The union of a sequence of countable sets is countable*
4. *The cartesian product of finitely many countable sets, is countable*

■

Summary last week (2/2)

- Infinite counting. Comparison via bijections.
- Diagonalisation

  None of the following are countable

  1. the set of infinite sequences over $\{a, b\}$
  2. functions $2^{\mathbb{N}}$; as infinite sequence is function $\mathbb{N} \to 2 = \{a, b\}$
  3. subsets $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$; by characteristic function $2^{\mathbb{N}}$
  4. reals $\mathbb{R}$; by sequence obtained by decimal expansion

Course themes

- directed and undirected graphs
- relations and functions
- orders and induction
- trees and dags
- finite and infinite counting
- elementary number theory
- Turing machines, algorithms, and complexity
- decidable and undecidable problem

Theorem (Schröder–Bernstein)

Let $f : A \to B$ and $g : B \to A$ be injective functions. Then there is a bijection $f' : A \to B$

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Example (Picture on the board/animation next slide)

Let $A = \mathbb{N}$, $B = \{a\}^*$, and $f : A \to B$, $g : B \to A$ be defined by:

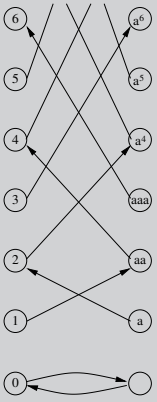
$$f(n) := a^{2n}$$
$$g(a^n) := 2n$$

f and g are injective; a bijection $f' : A \to B$ can be constructed from f,g by:

$$f'(n) := \begin{cases} \epsilon & \text{if } n = 0 \\ g^{-1}(n) = a^{\frac{n}{2}} & n \text{ has odd number of 2-factors} \\ f(n) = a^{2n} & \text{otherwise} \end{cases}$$

Animation of construction of bijection f' from injections f, g

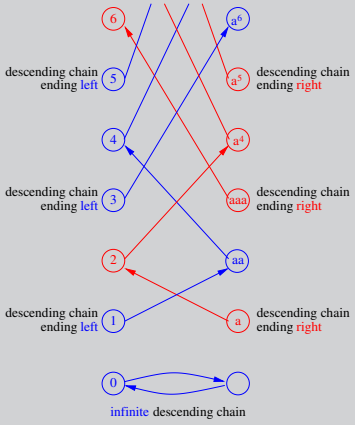
Example (Continued)



graph of f union g

Animation of construction of bijection f' from injections f, g

Example (Continued)



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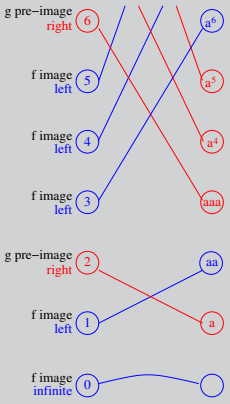

bijection f' by choosing g pre−image for nodes on chains ending right and f image for other nodes

Proof of Schröder–Bernstein theorem

Proof.

We construct $f' : A \to B$ and $g' : B \to A$ inverse to each other, as in animation.

- let $R = f \cup g$; viewed as relation on $A \uplus B$ (disjoint union)

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- define $f'(a)$ for $a \in A$ by cases on the colour of a:
  - **a)** $f'(a) := g^{-1}(a)$ (g pre-image if a is red; pre-image exists as a-chain ends on right)
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- verify f', g' inverse to each other. $f' \,;\, g'$ ($g' \,;\, f'$ analogous) by cases on colour $a \in A$:
  - **a)** $g'(f'(a)) = g'(g^{-1}(a)) = g(g^{-1}(a)) = a$, as $g^{-1}(a)$ is red if a is, being on same chain.
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Partially ordering sets up to equinumerosity

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$|M| := \{N \mid N \text{ equinumerous to } M\}$

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if $A, A' \in |M|$ and $B, B' \in |N|$, and injection $f : A \to B$, then exists injection $f' : A' \to B'$.

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for bijections $g : A' \to A$ and $g' : B \to B'$, composition $g \,;\, f \,;\, g' : A' \to B'$ is injection. ■

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$|\mathbb{N}| < |\mathbb{R}|$

Equivalence relations revisit

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Remark

An equivalence class contains all objects having the same property

Definition

$\{B_1, \ldots, B_n\}$ is a partition of M, if $B_1 \uplus \ldots \uplus B_n = M$ ($\uplus$ denotes unions disjoint)

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$\{\{000\}, \{001, 010, 100\}, \{011, 101, 110\}, \{111\}\}$ is a partition of $\mathbb{B}^3$

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(1) *Let P be a partition of M. Then is $\sim$ is an equivalence relation on M, such that*

$$x \sim y :\Leftrightarrow x \text{ and } y \text{ are in the same block of } P$$

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(3) *The functions $P \mapsto \sim$ in (1) and $\sim \mapsto P$ in (2) are inverse to each other*

From orders to equivalence relations

Lemma

if $\leq$ is a reflexive, transitive, then $\leq \cap \geq$ is induced equivalence relation.

Proof.

reflexivity, transitivity of $\leq \cap \geq$ hold by the same for $\leq$; symmetry by definition. ■

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Example

1. $\frac{n}{m} \leq \frac{n'}{m'}$ if $n \cdot m' \leq m \cdot n'$ induces the equivalence on (positive) fractions above
2. relating sets by injections induces equinumerosity
3. $\leq$ on natural numbers induces equality $=$

Elementary number theory: Euclid

Definition

- $d \in \mathbb{Z}$ is a divisor of $a \in \mathbb{Z}$, if there exists a $c \in \mathbb{Z}$ such that $a = c \cdot d$
- „d divides a“, „a is a multiple of d“ $d \mid a$
- the divisor $\pm 1, \pm a$ are called trivial divisors of a

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Definition

Let $a, b \in \mathbb{Z}$, $a, b \neq 0$

- The greatest common divisor $\gcd(a, b)$ of a and b divides a and b, and for all c such that $c \mid a$ and $c \mid b$, c divides $\gcd(a, b)$
- The least common multiple $\operatorname{lcm}(a, b)$ of a and b is a multiple of both a and b, and for all c such that $a \mid c$ and $b \mid c$, c is a multiple of $\operatorname{lcm}(a, b)$

Theorem

Let $a, b, c \in \mathbb{Z}$ *with* $a \neq 0$, $b \neq 0$ *and* $a \neq c \cdot b$; *then*

$$\gcd(a, b) = \gcd(|a|, |b|) \quad \textit{and} \quad \gcd(a, b) = \gcd(a - c \cdot b, b)$$

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- If an integer d divides a and b, then it also divides $a - c \cdot b$. Vice versa, if d divides $a - c \cdot b$ and b, then it also divides a.

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- If an integer d divides a and b, then it also divides $a - c \cdot b$. Vice versa, if d divides $a - c \cdot b$ and b, then it also divides a.
- the common divisors of a and b are the common divisors of $a - c \cdot b$ and b, and therefore they have the same greatest common divisors as well

Theorem (Euclidean algorithm for integers)

The greatest common divisor of non-zero integers can be computed as follows:

> *Replace the integers by their absolute values.*
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> *While the integers are distinct, repeat:*
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> > *Replace the larger of the two by the difference of the larger and the smaller.*
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Theorem (Variant)

Replace the integers by their absolute values.

While neither integer is a multiple of the other, repeat:

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- By the previous theorem, the greatest common divisors remain unchanged in each step of the algorithm, from which correctness follows
- Since the numbers remain positive in every iteration of the loop, and their maximum decreases by at least 1, the algorithm must terminate after finitely many steps

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Formal Proof.

claim: for all $n, m \geq 1$, `euclid m n` $= \gcd(m, n)$ where

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euclid m n = if m == n then m else if m > n  then euclid (m - n) n
else euclid m (n - m)
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Example

We have $\gcd(138, -48) = 6$, according to the first method:

$$\begin{aligned}\gcd(138, -48) &= \gcd(138, 48) = \gcd(90, 48) = \gcd(42, 48)\\ &= \gcd(42, 6) = \gcd(36, 6) = \gcd(30, 6)\\ &= \gcd(24, 6) = \gcd(18, 6) = \gcd(12, 6)\\ &= \gcd(6, 6) = 6\end{aligned}$$

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The second method yields

$$\gcd(138, -48) = \gcd(138, 48) = \gcd(42, 48) = \gcd(42, 6) = 6\,.$$

Theorem (Bézout's lemma)

Let a and b be non-zero integers. Then there exist natural numbers u and v with

$$u \cdot a + v \cdot b = \gcd(a, b)$$

which can be computed by the following algorithm

Set $A = (|a|, 1, 0)$ *and* $B = (|b|, 0, 1)$.
While B_1 *does not divide* A_1*, do:*
    Compute the integer quotient of A_1 *and* B_1.
    Set $C = B$.
    Set $B = A - q \cdot C$ *(componentwise)*
    Set $A = C$.
Set $u = sgn(a) \cdot B_2$ *and* $v = sgn(b) \cdot B_3$.

Proof.

- Let $T = (T_1, T_2, T_3)$ be a triple of integers and $(*)$ the property

$$T_1 = |a| \cdot T_2 + |b| \cdot T_3 \qquad (*)$$

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- If the triples A and B have the property $(*)$, then so do all triples $A - q \cdot B$ and $q \in \mathbb{Z}$.

Proof.

- Let $T = (T_1, T_2, T_3)$ be a triple of integers and $(*)$ the property

$$T_1 = |a| \cdot T_2 + |b| \cdot T_3 \qquad (*)$$

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- The first two triples in the algorithm have this property, hence all the subsequent triples have it as well. Restricting to the first components of triples the Euclidean algorithm is obtained. Therefore, we have for the final triples B

$$\gcd(a, b) = B_1 = |a| \cdot B_2 + |b| \cdot B_3 = \underbrace{(\operatorname{sgn}(a) \cdot B_2)}_{u} \cdot a + \underbrace{(\operatorname{sgn}(b) \cdot B_3)}_{v} \cdot b$$

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Example

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A	B	q
$(138, 1, 0)$	$(48, 0, 1)$	2
$(48, 0, 1)$	$(42, 1, -2)$	1
$(42, 1, -2)$	$(6, -1, 3)$	

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Let a and b be non-zero integers. Then

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Obviously,

$$m := \frac{|b|}{\gcd(a,b)} \cdot |a| = \frac{|a|}{\gcd(a,b)} \cdot |b|$$

is a multiple both of a and b, hence a common multiple. We show that m is the least common multiple of a and b. To that end, let z be an arbitrary positive common multiple of a and b. Then there are integers c, d with

$$z = c \cdot a \quad \text{and} \quad z = d \cdot b$$

Proof (continued).

By the previous theorem there exists integers u, v with

$$u \cdot a + v \cdot b = \gcd(a, b)$$

Hence

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so that z is a multiple of m. As $z, m > 0$, we have $\operatorname{sgn}(a \cdot b) \cdot (u \cdot d + v \cdot c) > 0$, hence $z \geqslant m$ and by $>$ being total, m is the least common multiple of a and b.

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Proof.

Let $c \in \mathbb{Z}$ with $c \cdot p = a \cdot b$. If p divides a, we are done. If p does not divide a, then $\gcd(a, p) = 1$, hence there are integers u and v such that $1 = u \cdot a + v \cdot p$; and therefore

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It suffices to show that the prime factors are unique. For a proof by contradiction, suppose $p_1 p_2 \cdots p_k$ and $q_1 q_2 \cdots q_\ell$ were distinct decompositions of a into prime factors.

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We show the claim by well-founded induction w.r.t. $<$.

Since p_1 divides the product $q_1 q_2 \cdots q_\ell$, we have by the previous theorem an index $j \in \{1, \ldots, \ell\}$ such that $p_1 = q_j$, $p_1 \geqslant 2$; from which it follows

$$p_2 \cdots p_k = \prod_{\substack{1 \leqslant i \leq \ell \\ i \neq j}} q_i$$

The claim follows by the IH. ∎