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Functional Programming

Week 11 – Lazy Evaluation, Infinite Lists

René Thiemann   Jonathan Bodemann   James Fox   Joshua Ocker   Daniel Rainer
Daniel Ranalter   Christian Sternagel

Department of Computer Science

Last Lecture

- `IO a` is type of I/O-actions with resulting type `a`
- `do`-blocks are used for sequential composition of I/O-actions
- clear separation between purely functional and I/O-code:
  - embed functional code into I/O: `return :: a -> IO a`
  - the other direction is not available: no function of type `IO a -> a`
- ghc compiles programs that provide `main :: IO ()` function in module `Main`
- example application: connect four
  - user-interface: I/O-code
  - game logic: purely functional

Monads

- bind (>>=), `return`, and **do**-notation are not restricted to I/O

- there exists a more general concept of monads

- example: also the Maybe-type is a monad

```
data Expr = Const Double | Div Expr Expr
eval :: Expr -> Maybe Double
eval (Const c) = return c
eval (Div expr1 expr2) = do
  x1 <- eval expr1
  x2 <- eval expr2
  if x2 == 0
    then Nothing
    else return (x1 / x2)
```

- monads won't be covered here, but they are the reason why the Haskell literature speaks about the I/O-monad

Evaluation Strategies

Pure Functions

- a function is pure if it always returns same result on same input
- pure functions are similar to mathematical functions
- examples of pure functions
    - addition
    - sort a list
    - . . .
- examples of non-pure functions
    - roll a dice
    - current time
    - position of cursor
    - . . .
- pure languages permit to define only pure functions
- Haskell is a pure language

Pure Functions and I/O

- even I/O is pure in Haskell
- consider `main = getLine >>= putStrLn . ("Hello " ++)`
- it seems that the result depends on user input, so is not pure
- however `main :: IO ()`, so the functional value of `main` is not what is entered and printed during execution, but the value is of type `IO ()`, i.e., a sequence of actions that are executed when running the program; and indeed this sequence is always the same:

  first read some input i and then print the string `"Hello $i$"`

- alternative argumentation: interpret type `IO a` a state transformer on the outside world, e.g., as a function of type `RealWorld -> (RealWorld, a)`
- remark: in the remainder of this lecture we will only consider purely functional programs without I/O

Evaluation Order

- there are several ways to evaluate expressions, consider `square x = x * x`

$$\text{square } (5 + 3)$$

$$(5 + 3) * (5 + 3) \qquad \text{square } 8$$

$$8 * (5 + 3) \qquad (5 + 3) * 8$$

$$8 * 8$$

$$64$$
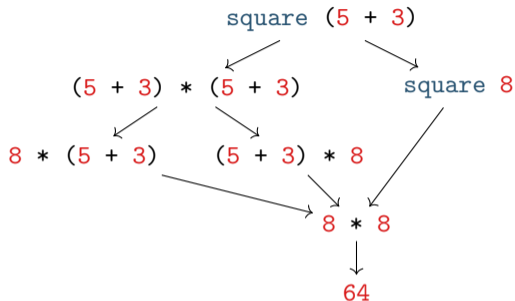
- in pure languages, the evaluation order has no impact on resulting normal form
- normal form: an expression that cannot be evaluated further, a result

Theorem

Whenever there are two (different) ways to evaluate a Haskell expression to normal form, then the resulting normal forms are identical.

Standard Evaluation Strategies

- each functional language fixes the evaluation order via some evaluation strategy
- three prominent evaluation strategies (expressions reprented as trees and dags)
    - call-by-value / strict / innermost: first evaluate arguments

    ```
                    square     square        *
                       |          |         / \
    square (5+3) =     +     =    8     =  8   8   =   64
                      / \
                     5   3
    ```

    - call-by-name / non-strict / outermost: directly replace function application by rhs

    ```
                    square        *             *           *
                       |         / \           / \         / \
    square (5+3) =     +     =  +   +    =    +   8   =   8   8   =   64
                      / \       /\  /\        /\
                     5   3     5 3 5 3       5 3
    ```

    - call-by-need / lazy evaluation: like call-by-name + sharing (dags = directed acyclic graphs)

    ```
                    square        *             *
                       |         ( )           ( )
    square (5+3) =     +     =    +       =      8      =   64
                      / \        / \
                     5   3      5   3
    ```

Evaluation Strategy of Haskell

- Haskell uses lazy evaluation with left-to-right argument order
- sharing is applied whenever a variable occurs multiple times
- example: consider definition `f x = g x + g (3 + 5) + x`
  - when evaluating `f (1 + 2) = g (1 + 2) + g (3 + 5) + (1 + 2)` the two occurrences of `1 + 2` are shared: they use the same variable `x`
  - when evaluating `f (3 + 5) = g (3 + 5) + g (3 + 5) + (3 + 5)` the two occurrences of `g (3 + 5)` are not shared: it was a coincidence that `x` was substituted by `3 + 5` and this equality is not detected at runtime
- there might be further sharing (depending on the compiler), e.g. sharing common subexpressions such as the expression `g x` in a function definition `f x = g x + h (g x)`
- argument evaluation within function invocation `f expr1 ... exprN` is mainly triggered by pattern matching, i.e., the process of finding the suitable defining equation `f pat1 ... patN = expr`, cf. slides 12 and 14 of week 3
- many builtin arithmetic functions will trigger evaluation of all arguments, e.g., `(0 :: Integer) * undefined` will result in error, and not in `0`

Evaluation Strategy and Termination

- consider the following Haskell script
  ```
  three :: Integer -> Integer
  three x = 3

  inf :: Integer
  inf = 1 + inf
  ```
- strict evaluation does not terminate, i.e., it will evaluate forever
  ```
  three inf = three (1 + inf) = three (1 + (1 + inf)) = ...
  ```
- non-strict and lazy evaluation are immediately done
  ```
  three inf = 3
  ```

Theorem

- if the evaluation of an expression terminates for some evaluation strategy, then it terminates using non-strict or lazy evaluation
- if the evaluation of an expression terminates using strict evaluation, then it terminates for every evaluation strategy

Comparison of Evaluation Strategies

- call-by-value
    - easy to understand
    - easy to implement
    - overhead in evaluating non-required expressions
    - used in many functional programming languages
- lazy evaluation
    - harder to understand
    - single evaluation step is more complicated to implement:
      pass arguments that are unevaluated expressions (thunks) instead of just values
    - overhead in computing with thunks
    - allows programmers to naturally define and work with infinite data
    - used in Haskell

Tail Recursion and Strict Evaluation

Different Kinds of Recursion

- a function calling itself is recursive
- functions that mutually call each other are mutually recursive

```
even n | n == 0    = True
       | otherwise = odd (n - 1)
odd n  | n == 0    = False
       | otherwise = even (n - 1)
```

- nested recursion: recursive calls inside recursive calls

```
ack n m | n == 0 = m + 1
        | m == 0 = ack (n - 1) 1
        | otherwise = ack (n - 1) (ack n (m - 1))
```

- linear recursion: at most one recursive call (per if-then-else branch)
  - `fib n | n >= 2 = fib (n - 1) + fib (n - 2)`   ✘
  - `length (x : xs) = 1 + length xs`   ✔
  - `f x = if even x then f (x ` + "`div`" + ` 2) else f (3 * x + 1)`   ✔

- tail recursion and guarded recursion will be discussed in more detail

Tail Recursion

- tail recursion is special form of linear recursion
- additional requirement
    - recursive function calls happen at the outermost level
    - however, they can be within an if-then-else
- examples
    - `length (x : xs) = 1 + length xs`   ✘
    - `f x = if even x then f (x `div` 2) else f (3 * x + 1)`   ✔
- advantage of tail recursion
    - no dangling function calls
    - can be evaluated as loop
    - space efficient

Example: Advantage of Tail Recursion

- linear but not tail recursive variant
  ```
  sumRec 0 = 0
  sumRec n = n + sumRec (n - 1)

    sumRec 5 = 5 + sumRec (5 - 1)
  = 5 + sumRec 4 = 5 + (4 + sumRec (4 - 1))
  = 5 + (4 + sumRec 3) = 5 + (4 + (3 + sumRec (3 - 1))) = ...
  = 5 + (4 + (3 + (2 + (1 + 0)))) = ... = 15   -- linear space
  ```
- tail recursive variant using accumulator to store intermediate results
  ```
  sumTr n = aux 0 n where
    aux acc 0 = acc
    aux acc n = aux (acc + n) (n - 1)

    sumTr 5
  = aux 0 5 = aux (0 + 5) (5 - 1)
  = aux 5 4 = aux (5 + 4) (4 - 1)
  = aux 9 3 = ... = 15
   -- constant space, implement as loop with two variables: acc and n
  ```

Problem of Tail Recursion using Lazy Evaluation

```
sumTr n = aux 0 n where
  aux acc 0 = acc
  aux acc n = aux (acc + n) (n - 1)
```

- example evaluation of `sumTr` on previous slide used call-by-value
- in lazy evaluation `acc` and `n` are only evaluated on demand
- causes linear memory consumption in `sumTr`

```
  sumTr 5                    -- with lazy evaluation
= aux 0 5
= aux (0 + 5) (5 - 1)
= aux (0 + 5) 4
= aux ((0 + 5) + 4) (4 - 1)
= ...
= aux (((((0 + 5) + 4) + 3) + 2) + 1) 0
= (((((0 + 5) + 4) + 3) + 2) + 1 = ... = 15
```

Enforcing Evaluation

- Haskell function to enforce evaluation: `seq :: a -> b -> b`
- evaluation of `seq x y` first evaluates `x` to WHNF and then returns `y`
- WHNF: weak head normal form
- expression `e` is in WHNF iff it has one of the following three shapes
  - `e = C expr1 ... exprN` for some constructor `C`   (constructor application)
  - `e = f expr1 ... exprN` if the defining equations of `f` have $M > N$ arguments, i.e., they
    are of the form `f pat1 ... patM = expr`   (too few arguments)
  - `e = \ pat1 ... patN -> expr`   (λ-abstraction)
- examples
  - in WHNF: `True`, `7.1`, `(5+1) : [1] ++ [2]`, `(:)`, `undefined : undefined`, `(++)`,
    `(++ undefined)`, `\ x -> undefined`
  - not in WHNF: `[1] ++ [2]`, `(\ x -> x + 1) (1 + 2)`, `undefined ++ undefined`
  - evaluation: `let x = 1 + 2 in seq x (f x)`
    `= seq (1 + 2) (f (1 + 2))`   `-- with 1 + 2 shared`
    `= seq 3 (f 3)`   `-- seq enforced evaluation of argument`
    `= f 3 = ...`   `-- evaluation of f 3 continues`

Example Application using `seq`

- solve memory problem in tail recursion by enforcing evaluation of accumulator

```
sumTrSeq n = aux 0 n where
  aux acc 0 = acc
  aux acc n = let accN = acc + n in seq accN (aux accN (n - 1))

  sumTrSeq 5
= aux 0 5
= let accN = 0 + 5 in seq accN (aux accN (5 - 1))
= seq (0 + 5) (aux (0 + 5) (5 - 1))                  -- 0 + 5 is shared
= seq 5 (aux 5 (5 - 1))                               -- and evaluated
= aux 5 (5 - 1)
= aux 5 4                             -- pattern matching triggers evaluation
= let accN = 5 + 4 in seq accN (aux accN (4 - 1))
= seq (5 + 4) (aux (5 + 4) (4 - 1))                   -- 5 + 4 is shared
= seq 9 (aux 9 (4 - 1))                               -- and evaluated
= aux 9 (4 - 1)                               -- same structure as above
= ... = 15                                    -- constant space
```

Enforcing Strict Evaluation . . . Continued

- besides `seq`, there are other options to enforce strict evaluation
- strict library functions like a strict version of `foldl`:
  `Data.List.foldl' :: (b -> a -> b) -> b -> [a] -> b`

  ```
  import Data.List
  length = foldl' (\ x _ -> x + 1) 0
  ```
- pattern matching with bang patterns to enforce evaluation, e.g.,
  `aux acc n = let !accN = acc + n in aux accN (n - 1)`
- strict datatypes
- see https://downloads.haskell.org/~ghc/9.2.5/docs/html/users_guide/exts/strict.html for further details

Lazy Evaluation and Infinite Lists

Guarded Recursion

- every recursive call is inside ("guarded by") a constructor
- also known as "tail recursion modulo cons"
- more important than tail recursion in Haskell
- allows the result to be consumed lazily – tail recursion provides the result only at the end
- examples
    - ```
      map f [] = []
      map f (x:xs) = f x : map f xs                                    ✔
      ```
    - ```
      reverse xs = revAux xs [] where
        revAux [] ys = ys                                             ✘
        revAux (x : xs) ys = revAux xs (x : ys)
      ```
    - ```
      enumFrom x = x : enumFrom (x + 1)                               ✔
      ```
- remarks on `enumFrom`
    - above definition is simplified, actual definition works for members of type class `Enum`, e.g., `Int`, `Char`, `Integer`, `Double`, ... and prevents overflows
    - syntactic sugar: `[x..] = enumFrom x`

Infinite Lists

- infinite lists $\sim$ sequences of elements (also known as streams)
- programming with infinite lists: producing and consuming elements of sequences one after another (e.g., with guarded recursion)
- example: `[x..] = x : [x + 1 ..]` generates infinite list
- in combination with lazy evaluation, infinite lists do not always cause non-termination
- examples

```
  take 2 [7..]
= take 2 (7 : [8..])
= 7 : take 1 [8..]
= 7 : 8 : take 0 [9..]
= [7, 8]

  takeWhile (< 95) $ map (\ x -> x * x) [0..]
= ... = [0,1,4,9,16,25,36,49,64,81]

  filter (< 100)  $ map (\ x -> x * x) [0..]
= ... = [0,1,4,9,16,25,36,49,64,81   -- interrupted
```

Laziness and Infinite Data Structures Facilitate Modularity

- separation of concerns
    - write small functions with specific tasks
    - use potentially infinite data structures
- example: find index of first list element satisfying predicate
    - function `firstIndex :: (a -> Bool) -> [a] -> Int`
    - in Haskell
      `firstIndex p = fst . head . filter (p . snd) . zip [0..]`
    - (lazy) evaluation (without showing expansion of (.) and ($))
      ```
        firstIndex (== 1) [1..9]
      = fst . head . filter ((== 1) . snd) $ zip [0..] [1..9]
      = fst . head . filter ((== 1) . snd) $ (0,1) : zip [1..] [2..9]
      = fst . head $ (0,1) : filter ((== 1) . snd) $ zip [1..] [2..9]
      = fst (0,1)
      = 0
      ```
    - without laziness several complete list traversals are required when using library functions
      (e.g., computation of length and addition of indices)
    - remark: `firstIndex` works for arbitrary lists as input: finite and infinite

Sieve of Eratosthenes

- goal: generate list of all prime numbers
- algorithm
    1. start with list of all natural numbers (from 2 on)
    2. mark first element x as prime
    3. remove all multiples of x
    4. go to Step 2
- in Haskell
  ```haskell
  primes :: [Integer]
  primes = sieve [2..] where
    sieve (x : xs) = x : sieve (filter (\ y -> y `mod` x /= 0) xs)

  > take 1000 primes          -- the first 1000 primes
  > takeWhile (< 1000) primes -- all primes below 1000
  ```

Summary

- in pure functional languages such as Haskell the result does not depend on the evaluation strategy
- different kinds of recursion
- tail recursion is usually efficient as it can be implemented as loop
- `seq` can be used to enforce strict evaluation (in particular of accumulators)
- lazy evaluation allows modeling of infinite lists
- guarded recursion is important for algorithms on infinite lists
- infinite lists permit to naturally formulate several algorithms
  (without having to take care of boundary conditions)