universität innsbruck

Recent Advances in Formalising Modern Number Theory

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Computational Logic Group

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- Large library of mathematics (especially analysis)

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Current Work

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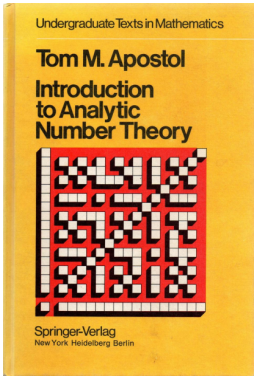

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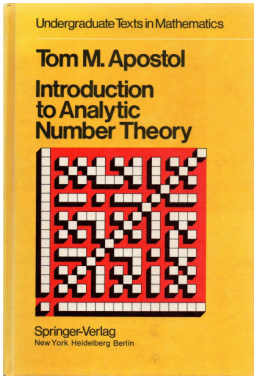

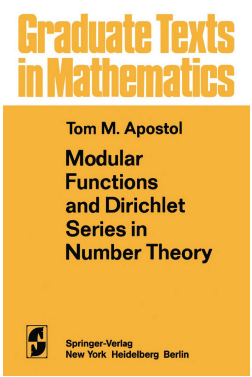

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Given two complex numbers z_1, z_2 with $z_1/z_2 \notin \mathbb{Q}$, the *complex lattice* $\Lambda(z_1, z_2)$ is the set of all complex numbers of the form $az_1 + bz_2$ for $a, b \in \mathbb{Z}$.

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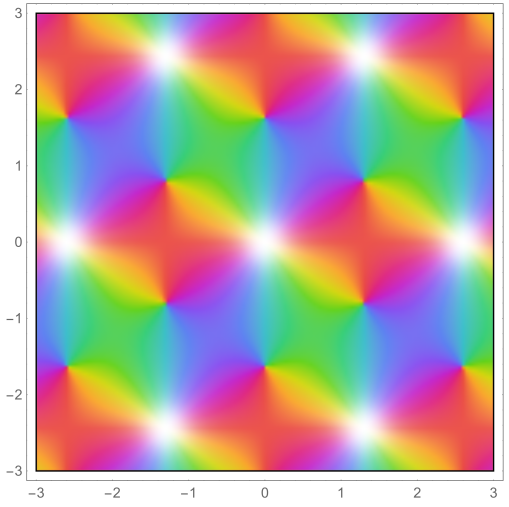
Fact

The Weierstraß function and its derivative are elliptic:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \qquad \wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$$

All others can be expressed in terms of $\wp$ and $\wp'$.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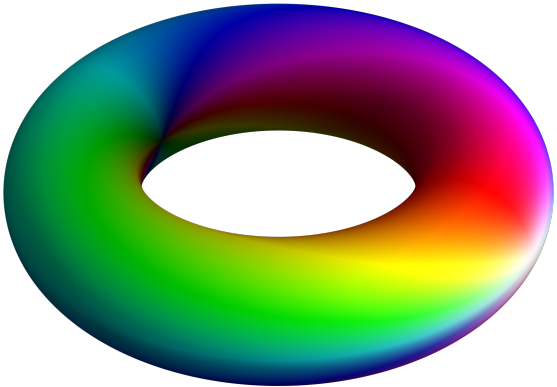

Weierstraß $\wp$ and the Eisenstein Series

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$\wp$ has the following series expansion at $z = 0$:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (k+1) G_{k+2} z^k \qquad \text{where } G_k = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^k} \text{ for } n \geq 3$$

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$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$$

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Note: $y^2 = 4x^3 - ax - b$ is an elliptic curve.
$z \mapsto (\wp(z), \wp'(z))$ maps points on a torus $\mathbb{C}/\Lambda$ to points on a complex elliptic curve.

Example: Differential Equation of $\wp$

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```
theorem weierstrass_fun_ODE1':
  assumes "z ∉ Λ"
  shows   "℘' z ^ 2 = 4 * ℘ z ^ 3 - 60 * G₄ * ℘ z - 140 * G₆"
```

Example: Fourier Expansion of G_k

$$G_k(z) = 2(\zeta(k) + \frac{(2i\pi)^k}{(k-1)!} \sum_{n=0}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z})$$

```
lemma Eisenstein_G_fourier_expansion:
  assumes "Im z > 0" and "k > 2" and "even k"
  shows "Eisenstein_G k z =
           2 * (zeta k + (2*i*pi)^k / fact (k-1) *
                  (∑n. divisor_sigma (k-1) n * exp (2*n*pi*i*z)))"
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$$(2n+3)(n-2)b_n = 3\sum_{k=1}^{n-2} b(k)b(n-k-1)$$

where $b_n = (2n+1)G_{2n+2}$ *for any* $n \geq 3$.

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G_4 and G_6 are the important ones.

Example:

```
theorem eisenstein_series_as_polynomials:
  defines "b ≡ λn. (2*n + 1) * (G (2*n + 2))"
  assumes "n ≥ 3"
  shows   "(2 * n + 3) * (n - 2) * b n = 3 * (∑i=1..n-2. b i * b (n - i - 1))"
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Observations:

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It can be shown that $J(\Lambda_1) = J(\Lambda_2)$ iff Λ_1 and Λ_2 are homothetic.

Modular Forms

Functions like G_k, Δ, J that are smooth maps $\Lambda(\mathbb{C}) \to \mathbb{C}$ and satisfy an equation of the form $f(c\Lambda) = c^k f(\Lambda)$ for some weight $k \in \mathbb{Z}$ are called *modular forms*.

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Bottom line: invariant under modular transformations $z \mapsto z + 1$ and $z \mapsto -\frac{1}{z}$ and doesn't do anything "crazy" in $\mathbb{H} \cup \{i\infty\}$.

Modular Transformations and Fundamental Domain

The transformations $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ with function composition form a group – the so-called modular group.

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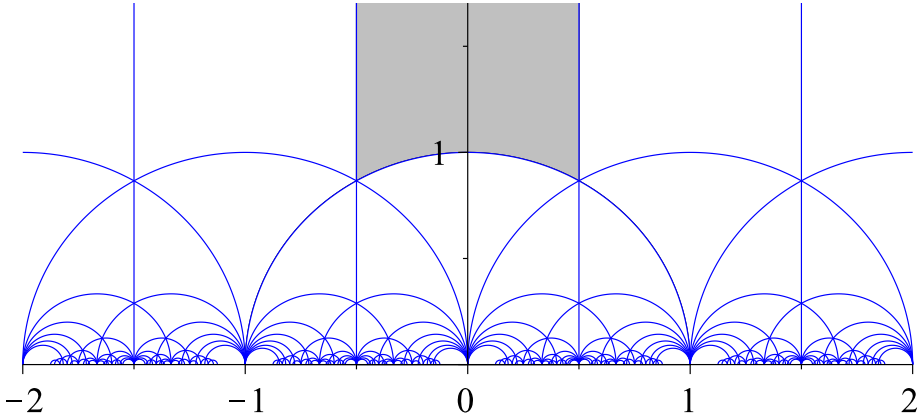
Each point in $\mathbb{H}$ has a canonical representative in the fundamental domain $|z| \geq 1$, $|\mathrm{Re}(z)| \leq \frac{1}{2}$.

The Fundamental Domain



Source: https://commons.wikimedia.org/wiki/File:ModularGroup-FundamentalDomain.svg

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Consequence: entire modular forms are linear combinations of generators $G_{k-12r}\Delta^r$

Example: Valence Formula

```
theorem valence_formula:
   "(∑z∈𝒟Γ'∩(poles∪zeros)-{i,ϱ}. zorder f z) +
       zorder f ϱ / 3 + zorder f i / 2 + zorder_at_cusp = k / 12"
```

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Some Examples

Any modular function is a rational function of Klein's *J* invariant:

```
theorem (in modular_function') in_terms_of_Klein_j:
  obtains p q :: "complex poly"
  where "q ≠ 0" "⋀z. Im z > 0 ⟹ z ∉ poles ⟹
                        f z = poly p (Klein_j z) / poly q (Klein_j z)"
```

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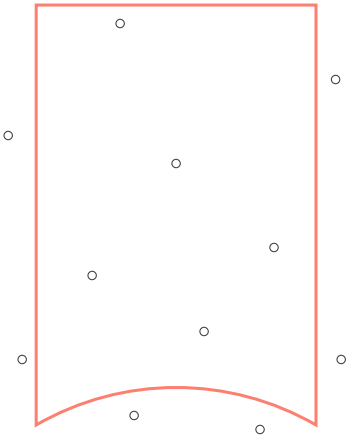
The proof in the book is not much longer than this sketch here, but full of subtleties that make formalisation very tricky.

Proving the Valence Formula



We want to count the zeros/poles inside the fundamental region.

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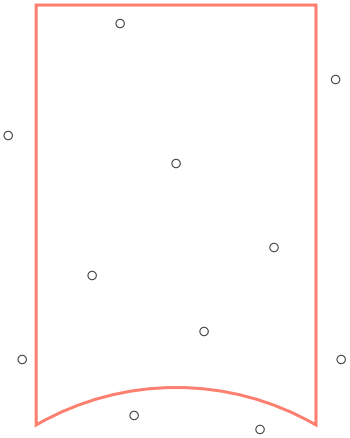

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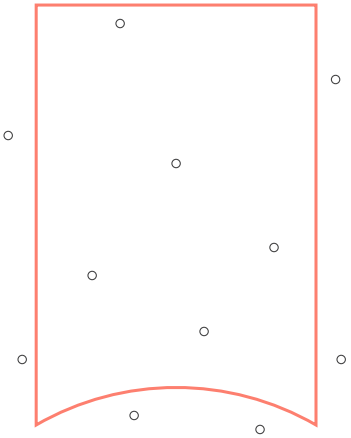

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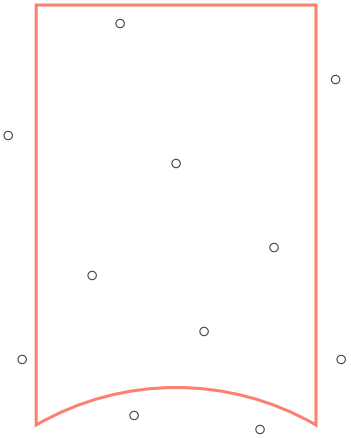

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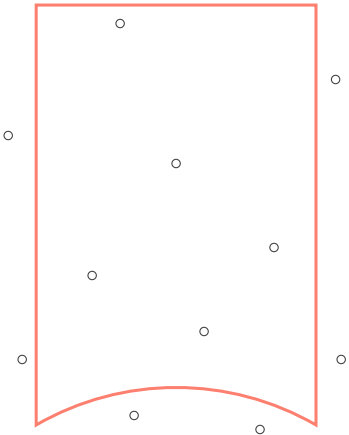

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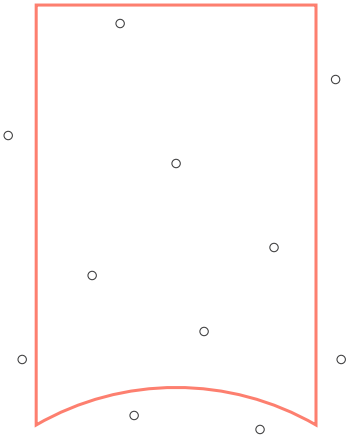

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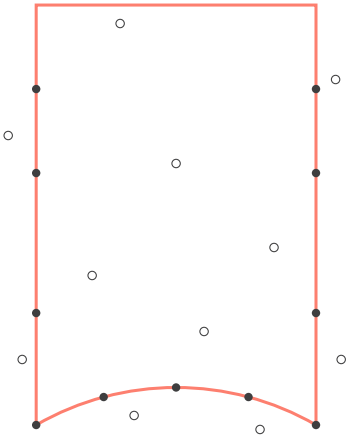

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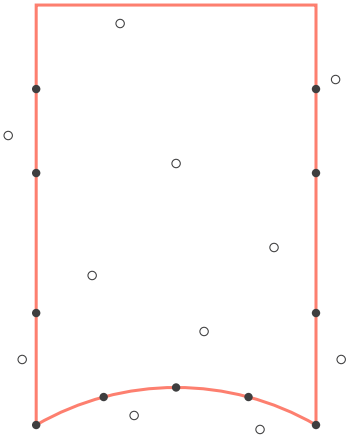

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Solution: Deform γ with small arcs of radius $\varepsilon \to 0$.

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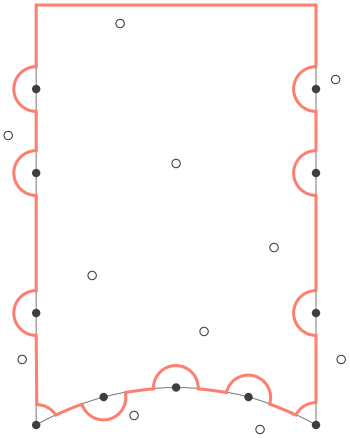

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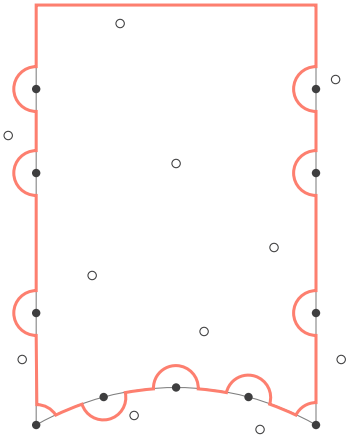

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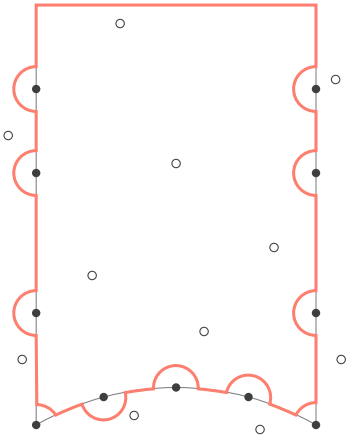

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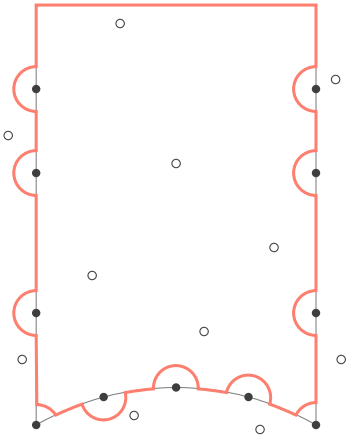

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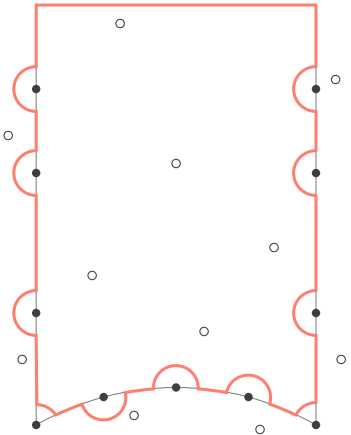

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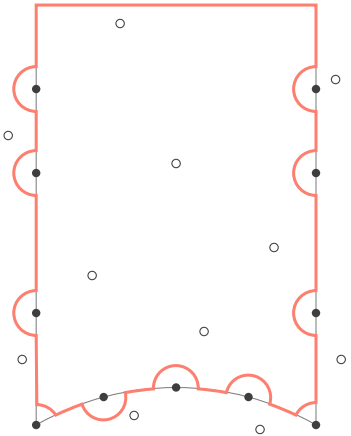

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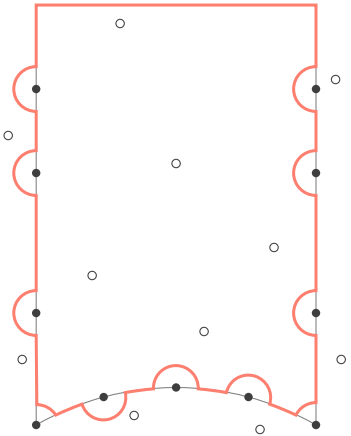

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This is already tedious for “easy” curves and completely unfeasible for something like this.

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But how to show $\gamma \underset{L/R}{\approx} \gamma_\varepsilon$?

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Basic Avoidance Patterns

Straight line: 

Basic Avoidance Patterns

Straight line: 

Circular arc: 

Basic Avoidance Patterns

Straight line: 

Circular arc: 

Straight line corner: 

Basic Avoidance Patterns

Straight line:

Circular arc:

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Line/arc corner:

Basic Avoidance Patterns

Straight line:

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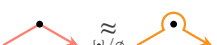

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These rules have only trivial side conditions and are thus very easy to apply.

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- But it also clarifies many aspects.