SAT and SMT Solving | WS 2022 | LVA 703147

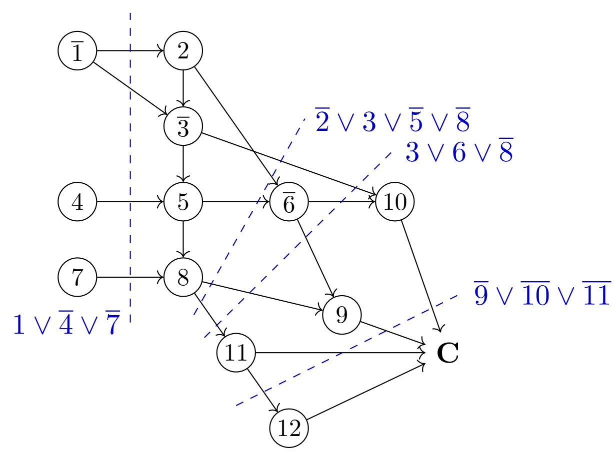
Solutions to Test 1 | December 2, 2022

$\boxed{1}$ We consider the formula

$$(1 \vee 2) \wedge (1 \vee \overline{2} \vee \overline{3}) \wedge (3 \vee \overline{4} \vee 5) \wedge (\overline{2} \vee \overline{5} \vee \overline{6}) \wedge (\overline{5} \vee \overline{7} \vee 8) \wedge (6 \vee \overline{8} \vee 9) \wedge (3 \vee 6 \vee 10) \wedge (\overline{8} \vee 11) \wedge (\overline{11} \vee 12) \wedge (\overline{9} \vee \overline{10} \vee \overline{11} \vee \overline{12})$$

and a DPLL inference sequence that reached the state $\overline{1}^d\, 2\, \overline{3}\, 4^d\, 5\, \overline{6}\, 10\, 7^d\, 8\, 9\, 11\, 12$.

(a) The implication graph looks as follows:



Three possible cuts with the implied clauses are indicated. The UIPs are 7 and 8.

(b) The clause $\overline{9} \vee \overline{10} \vee \overline{11}$ has minimal size among the clauses derived from cuts. Using resolution it can be derived as follows:
The conflict clause is $\overline{9} \vee \overline{10} \vee \overline{11} \vee \overline{12}$, its literal whose complement was assigned last is $\overline{12}$. The clause responsible for this assignment is $\overline{1}1 \vee 12$. We thus resolve

$$\frac{\overline{9} \vee \overline{10} \vee \overline{11} \vee \overline{12} \qquad \overline{1}1 \vee 12}{\overline{9} \vee \overline{10} \vee \overline{11}}$$

$\boxed{2}$ Consider the following EUF formula:

$$\mathsf{a} = \mathsf{b} \wedge \mathsf{c} = \mathsf{g}(\mathsf{a}) \wedge \mathsf{f}(\mathsf{a},\mathsf{a}) = \mathsf{c} \wedge \mathsf{f}(\mathsf{b},\mathsf{b}) = \mathsf{f}(\mathsf{c},\mathsf{b}) \wedge \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b}) \neq \mathsf{g}(\mathsf{a})$$

As the formula contains only one inequality, one has to check whether the equations (call them E) imply the inequality.
One can start by putting all subterms into different sets:

1. $\{\mathsf{a}\}$	2. $\{\mathsf{b}\}$	3. $\{\mathsf{c}\}$	4. $\{\mathsf{g}(\mathsf{a})\}$
5. $\{\mathsf{f}(\mathsf{a},\mathsf{a})\}$	6. $\{\mathsf{f}(\mathsf{b},\mathsf{b})\}$	7. $\{\mathsf{f}(\mathsf{c},\mathsf{b})\}$	8. $\{\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})\}$

After merging sets according to the equations, one gets

$$1.\quad \{\mathsf{a},\mathsf{b}\} \qquad 3.\quad \{\mathsf{c},\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a},\mathsf{a})\} \qquad 6.\quad \{\mathsf{f}(\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{c},\mathsf{b})\} \qquad 8.\quad \{\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})\}$$

One has to merge sets 3 and 6 because a and b are in the same set, so also $\mathsf{f}(\mathsf{a},\mathsf{a})$ and $\mathsf{f}(\mathsf{b},\mathsf{b})$ must be in the same set:

$$1.\quad \{\mathsf{a},\mathsf{b}\} \qquad 3.\quad \{\mathsf{c},\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{f}(\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{c},\mathsf{b})\} \qquad 8.\quad \{\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})\}$$

Now, since c and $\mathsf{g}(\mathsf{a})$ are in the same set, also $\mathsf{f}(\mathsf{c},\mathsf{b})$ and $\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})$ must be in the same set, so sets 3 and 8 can be merged:

$$1.\quad \{\mathsf{a},\mathsf{b}\} \qquad 3.\quad \{\mathsf{c},\mathsf{g}(\mathsf{a}),\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{f}(\mathsf{b},\mathsf{b}),\mathsf{f}(\mathsf{c},\mathsf{b}),\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b})\}$$

Thus $E \models_{EUF} \mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{b}) = \mathsf{g}(\mathsf{a})$, so the formula is unsatisfiable.

$\boxed{3}$ In the given formula, one can substitute literals as follows:

$$\underbrace{\mathsf{a}=\mathsf{b}}_{1} \wedge \underbrace{\mathsf{c}=\mathsf{d}}_{2} \wedge (\underbrace{\mathsf{b}\neq\mathsf{c}}_{\overline{3}} \vee \underbrace{\mathsf{b}=\mathsf{e}}_{4}) \wedge (\underbrace{\mathsf{a}\neq\mathsf{d}}_{\overline{5}} \vee \underbrace{\mathsf{b}\neq\mathsf{e}}_{\overline{4}}) \wedge (\underbrace{\mathsf{b}=\mathsf{c}}_{3} \vee \underbrace{\mathsf{c}=\mathsf{a}}_{6})$$

(a) With the above substitutions, the propositional skeleton is

$$1 \wedge 2 \wedge (\overline{3} \vee 4) \wedge (\overline{5} \vee \overline{4}) \wedge (3 \vee 6).$$

(b) We apply DPLL(T) as follows:

$$\begin{array}{lrl}
 & \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6 & \\
\Longrightarrow^{+} & 1\,2 \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6 & \text{unit propagate} \\
\Longrightarrow & 1\,2\,3^{d} \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6 & \text{decide} \\
\Longrightarrow & 1\,2\,3^{d}\,4 \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6 & \text{unit propagate} \\
\Longrightarrow & 1\,2\,3^{d}\,4\,\overline{5} \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6 & \text{unit propagate}
\end{array}$$

At this point we check the assignment in the theory of equality, which is unsatisfiable, because 1, 2, 3, and $\overline{5}$ form a contradictory cycle:

$$\mathsf{a} - - - \mathsf{b} - - - \mathsf{c} - - - \mathsf{d}$$

So we can T-learn the clause $\overline{1} \vee \overline{2} \vee \overline{3} \vee 5$, and afterwards T-backjump (actually, only backtrack):

$$\begin{array}{lrl}
\Longrightarrow & 1\,2\,3^{d}\,4\,\overline{5} \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6, \overline{1}\vee\overline{2}\vee\overline{3}\vee 5 & T\text{-learn} \\
\Longrightarrow & 1\,2\,\overline{3} \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6, \overline{1}\vee\overline{2}\vee\overline{3}\vee 5 & T\text{-backjump} \\
\Longrightarrow & 1\,2\,\overline{3}\,6 \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6, \overline{1}\vee\overline{2}\vee\overline{3}\vee 5 & \text{unit propagate}
\end{array}$$

We could now decide further, but actually we can already check EQ-consistency of this assignment, and it is inconsistent because 1, $\overline{3}$ and 6 form a cycle as well:

$$\mathsf{b} - - - \mathsf{a} - - - \mathsf{c}$$

So we can T-learn the clause $\overline{1} \vee \overline{3} \vee 6$, which causes DPLL($T$) to fail, so that the formula is unsatisfiable.

$$\begin{array}{lrl}
\Longrightarrow & 1\,2\,\overline{3}\,6 \parallel 1,\ 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{4},\ 3\vee 6, \overline{1}\vee\overline{2}\vee\overline{3}\vee 5,\ \overline{1}\vee\overline{3}\vee 6 & T\text{-learn} \\
\Longrightarrow & \mathsf{FailState} & \text{fail}
\end{array}$$

$\boxed{4}$ We have the formula

$$(\neg x \vee \neg y) \wedge \neg x \wedge (x \vee y) \wedge (x \vee \neg y \vee z) \wedge (x \vee z) \wedge (x \vee \neg y) \wedge \neg z.$$

(a) The assignment $v(x) = T$, $v(y) = v(z) = F$ satisfies all clauses except one. Since the formula is unsatisfiable, no assignment can satisfy all seven clauses, thus $\mathrm{maxSAT}(\varphi) = 6$ and $\mathrm{minUNSAT}(\varphi) = 1$.

(b) One can check that all combinations of two clauses of φ are satisfiable. So $\{\neg x, \neg z, x \vee z\}$ is a smallest unsatisfiable core of φ.