universität innsbruck

SAT and SMT Solving

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lecture 1
WS 2022

Important Information

- LVA 703147 (VU3)
- http://cl-informatik.uibk.ac.at/teaching/ws22/satsmt/

Time and Place

VU	Friday	14:15 – 17:00	SR12
PS	Friday	16:00 – 16:45	SR12

Grading

- 65% weekly exercises
- 35% tests on 2 December 2022 and 3 February 2023 (one hour each)
- attendence required

Exercises

- 10 points per week
- indicate solved exercises before Friday 10:00 in OLAT, submit solutions

Questions, Comments, Suggestions

- sarah.winkler@uibk.ac.at
- OLAT

Outline

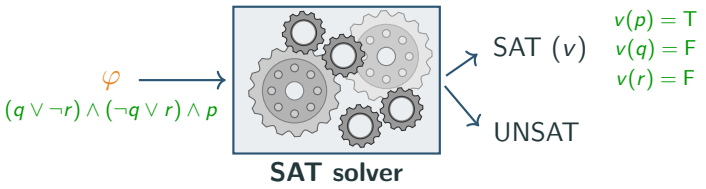
Outline

SAT Solving

input: propositional formula φ

output: SAT + valuation v such that $v(\varphi) = T$ if φ satisfiable

UNSAT otherwise



Terminology

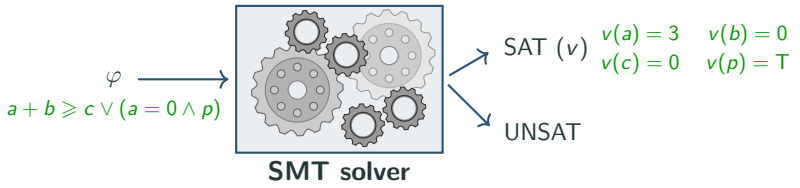
- decision problem P is problem with answer yes or no
- SAT encoding of decision problem P is propositional formula φ_P such that
  answer to P is yes $\iff$ φ_P is satisfiable

SMT Solving

input: formula φ involving theory T

output: SAT + valuation v such that $v(\varphi) = T$ if φ is T-satisfiable

UNSAT otherwise



Example (Theories)

- arithmetic $2a + b \geqslant c \vee (a = 0 \wedge p)$
- uninterpreted functions $f(x, y) \neq f(y, x) \wedge g(f(x, x)) = g(y)$
- bit vectors $((\text{zext}_{32}\ a_8) + b_{32}) \times c_{32} >_u 0_{32}$

Terminology

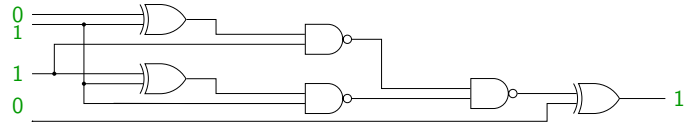
- SMT encoding over theory T of decision problem P is formula φ_P such that
  answer to P is yes $\iff$ φ_P is satisfiable

Application 1: Hardware Verification

Problem

- errors in hardware chips are costly (Intel paid $475 million for FDIV bug)
- testing is not enough to guarantee desired behavior

Example (Formal Circuit Model)



SAT Encoding

- variables for input and output
- SAT formulas for implemented behavior and expected behavior (specification)
- check for equivalence

Impact

- ensured correctness, more reliable hardware components (formal verification)
- manufacturers rely on SAT-based verification since beginning of 2000s
  e.g., Intel Core i7 implements over 2700 distinct verified micro-instructions

Application 2: Driving License Test

Problem

Austrian driving license test consists of 80 questions out of 1500 such that the following conditions are satisfied:

- 30 "main questions" with 3 sub-questions each
- at least 12 main questions must be about crossroads
- at least 12 main questions must have pictures
- at least 5 "hard", "medium", and "easy" main questions

▶ how can software find valid question set?

SAT encoding

- variables q_i for $1 \leqslant i \leqslant 1500$
- idea: valuation v sets $v(q_i) = \text{T}$ if question i is included, $v(q_i) = \text{F}$ otherwise

▶ $\sum_{i \in Q_{\text{xroads}}} q_i \geqslant 12$ ▶ $\sum_{i \in Q_{\text{pictures}}} q_i \geqslant 12$ ▶ $\sum_{i \in Q_{\text{hard}}} q_i \geqslant 5$ ▶ ...

Result

easy generation of valid question sets (with some random preselection)

Application 3: Pythagorean Triples

Problem
Can one color all natural numbers with two colors such that whenever $x^2 + y^2 = z^2$ not all of x, y, and z have same color?

Example

$3^2 + 4^2 = 5^2$ $\quad$ $5^2 + 12^2 = 13^2$

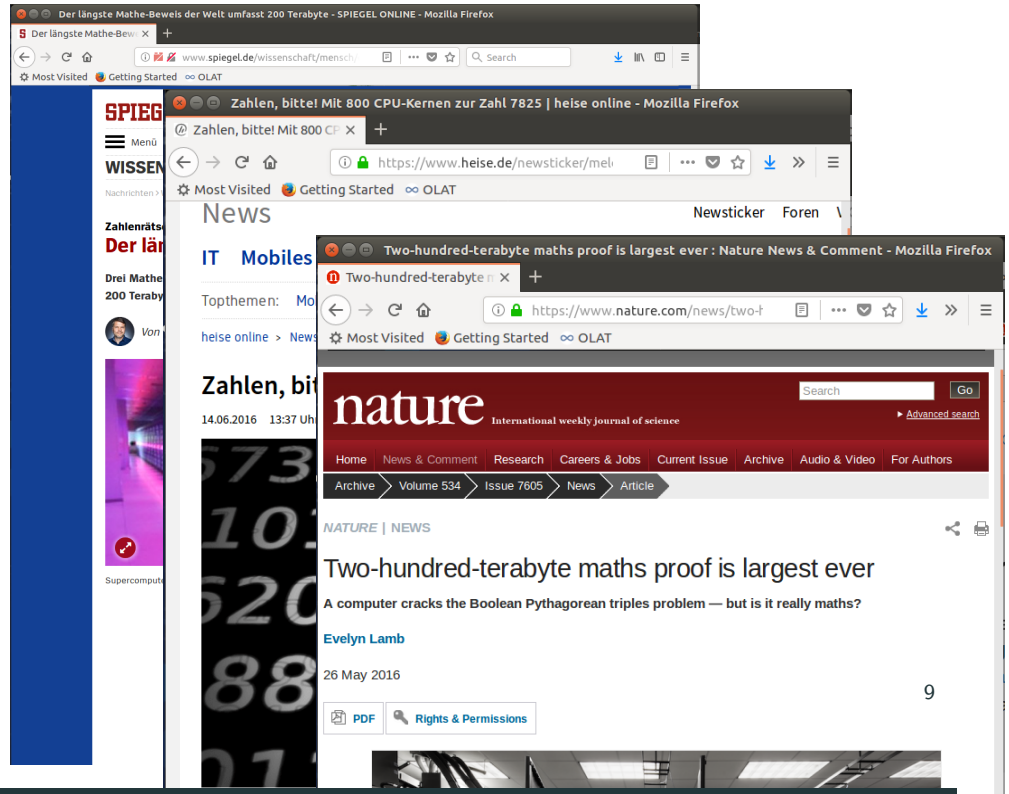
(a)	1	2	3	4	5	6	7	8	9	10	11	12	13	...	✓
(b)	1	2	3	4	5	6	7	8	9	10	11	12	13	...	✗

SAT encoding

- variables x_i for $1 \leqslant i \leqslant n$ such that x_i becomes true iff it is colored red
- SAT encoding: for all $a^2 + b^2 = c^2$ include $(x_a \vee x_b \vee x_c) \wedge (\bar{x}_a \vee \bar{x}_b \vee \bar{x}_c)$
  (+ symmetry breaking, simplification, heuristics)

Result: No. Coloring exists only up to 7,825.

1000s of variables, solving time 2 days with 800 processors, 200 TB of proof



Application 4: Tournament Scheduling

Problem: Round Robin scheduling
Schedule sports league tournament for n teams, p periods of $n-1$ rounds each
(+ venue restrictions, break restrictions, ...)

Example (Österreichische Fußball-Bundesliga)
10 teams play in 4 periods (9 rounds each),
periods 1 & 2 and 3 & 4 mirrored

(Part of) SAT encoding

- variable x_{ijpr} is true if team i plays team j at home in period p, round r
- $\bigwedge_{i,p,r} \bigvee_{j \neq i} (x_{ijpr} \vee x_{jipr})$ each team plays in every round

  $\bigwedge_{i,p,r} \bigwedge_{j \neq i} \bigwedge_{k \neq i \wedge k \neq j} (x_{ijpr} \rightarrow \neg(x_{ikpr} \vee x_{kipr}))$ each team plays at most once in every round

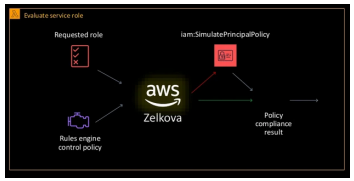
  $\bigwedge_{i,j,r} (x_{ij1r} \rightarrow x_{ji2r}) \wedge (x_{ij3r} \rightarrow x_{ji4r})$ mirror rounds 1& 2 and 3& 4

Result
SAT scheduling is 100x faster than previous industrial scheduling tools

Application 5: Policy Verification in AWS Zelkova

Problem

- in Amazon web services, users define complex access policies for services
- users want to check whether
  - policy allows agent X to do action Y
  - policy A is more/less restrictive, or equivalent to, policy B
- security critical
- should be checked automatically



SMT encoding

- using string and bitvector variables, and reasoning emulating regular expressions
- policy is encoded as $\bigvee_{S \in Allow}[S] \wedge \neg \bigvee_{S \in Deny}[S]$
- where for each statement S,
  $[S] := (\bigvee_{v \in P(S)} p{=}v) \wedge (\bigvee_{v \in A(S)} a{=}v) \wedge (\bigvee_{v \in R(S)} r{=}v) \wedge (\bigvee_{O \in C(S)}[O])$

Result

- Zelkova is invoked tens of millions of times per day
- latency in magnitude of milliseconds

Application 6: Network Verification in Microsoft Azure

Problem

- Microsoft Azure data centers must ensure cloud contracts:
  network access restrictions, forwarding tables, Border Gateway Protocol policies
- routing configuration should satisfy contracts – but routing tables change fast!

▶ cloud contracts should be verified automatically

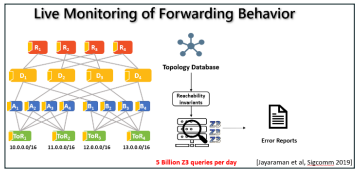
SMT encoding in SecGuru tool

- model network configuration as formula

$$
\begin{array}{l}
Router \equiv \\
\quad \textbf{if } \ldots \\
\quad \textbf{if } dst = 10.91.114.128/25 \textbf{ then } n_3 \vee n_6 \vee n_7 \textbf{ else} \\
\quad \textbf{if } dst = 10.91.114.0/25 \textbf{ then } n_3 \vee n_4 \vee n_5 \vee n_6 \textbf{ else} \\
\quad n_1 \vee n_2
\end{array}
$$

- express cloud contracts as formulas

$$\neg Cluster(dst) \wedge Router(dst) \Rightarrow \bigvee_n RouterAbove(n)$$



- assert that configuration does not satisfy contract: if satisfiable, found bug!

Result

- Azure uses billions of SMT queries for network verification every day

Hall of Fame

Herbrand Award 2019

Nikolaj Bjørner and **Leonardo de Moura**
"for their contributions to SMT solving, including its theory, implementation, and application to a wide range of academic and industrial needs"

CAV Award 2021

For pioneering contributions to the foundations of the theory and practice of satisfiability modulo theories (SMT).

Outline

Contents

Part 1: SAT

DPLL, conflict analysis, CDCL, 2-watched literals, heuristics, unsatisfiable cores, maxSAT, symmetry breaking

Part 2: SMT

DPLL(T), eager vs lazy, T-propagation, Nelson-Oppen combination, maxSMT

Part 3: Theory Solving

- equality with uninterpreted functions (congruence closure, conflict analysis)
- linear real arithmetic (simplex algorithm)
- arrays (reduction to EUF, lemmas on demand)
- bit vectors (bit blasting, preprocessing)

Practice

SAT solvers, SMT solvers, encoding, DIMACS, SMT-LIB, model checking

Outline

Propositional Logic Revisited

Concepts

- literal
- formula
- assignment
- satisfiability and validity
- negation normal form (NNF)
- conjunctive normal form (CNF)
- disjunctive normal form (DNF)

Definition (Propositional Logic: Syntax)

propositional formulas are built form

▶ atoms	p, q, r, p_1, p_2, ...	
▶ constants	$\bot$, $\top$	
▶ negation	$\neg p$	"not p"
▶ conjunction	$p \wedge q$	"p and q"
▶ disjunction	$p \vee q$	"p or q"
▶ implication	$p \rightarrow q$	"if p then q holds"
▶ equivalence	$p \leftrightarrow q$	"p if and only if q"

according to the BNF grammar

$$\varphi ::= p \mid \bot \mid \top \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi)$$

Conventions

- binding precedence $\neg \;>\; \wedge \;>\; \vee \;>\; \rightarrow, \leftrightarrow$
- omit outer parantheses
- $\rightarrow$, $\wedge$, $\vee$ are right-associative: $p \rightarrow q \rightarrow r$ denotes $p \rightarrow (q \rightarrow r)$

Definition (Propositional Logic: Semantics)

- valuation (truth assignment) is mapping $v : \{p, q, r, \dots\} \rightarrow \{\mathsf{F}, \mathsf{T}\}$ from atoms to truth values
- extension to formulas:

$$v(\bot) = \mathsf{F} \qquad v(\top) = \mathsf{T}$$

$$v(\varphi \wedge \psi) = \begin{cases} \mathsf{T} & \text{if } v(\varphi) = v(\psi) = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases} \qquad v(\neg\varphi) = \begin{cases} \mathsf{T} & \text{if } v(\varphi) = \mathsf{F} \\ \mathsf{F} & \text{if } v(\varphi) = \mathsf{T} \end{cases}$$

$$v(\varphi \vee \psi) = \begin{cases} \mathsf{F} & \text{if } v(\varphi) = v(\psi) = \mathsf{F} \\ \mathsf{T} & \text{otherwise} \end{cases} \qquad v(\varphi \leftrightarrow \psi) = \begin{cases} \mathsf{T} & \text{if } v(\varphi) = v(\psi) \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$v(\varphi \rightarrow \psi) = \begin{cases} \mathsf{F} & \text{if } v(\varphi) = \mathsf{T},\ v(\psi) = \mathsf{F} \\ \mathsf{T} & \text{otherwise} \end{cases}$$

Definitions

- formula φ is satisfiable if $v(\varphi) = \mathsf{T}$ for some valuation v
- formula φ is valid if $v(\varphi) = \mathsf{T}$ for every valuation v
- semantic entailment $\varphi_1, \ldots, \varphi_n \vDash \psi$
  if $v(\psi) = \mathsf{T}$ whenever $v(\varphi_1) = v(\varphi_2) = \cdots = v(\varphi_n) = \mathsf{T}$
- formulas φ and ψ are equivalent ($\varphi \equiv \psi$) if $v(\varphi) = v(\psi)$ for every valuation v
- formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \iff \psi \text{ is satisfiable}$$

Theorem

formula φ is unsatisfiable if and only if $\neg\varphi$ is valid

Theorem

satisfiability and validity are decidable

Proof.

Check all assignments (for n variables, 2^n possibilities). ■

Definition (Literal)

- literal is atom p or negation of atom $\neg p$
- literals l_1 and l_2 are complementary if $l_1 = \neg l_2$ or $l_2 = \neg l_1$
- write l^c for complementary literal of l

Definitions

- negation normal form (NNF) if formula with negation only applied to atoms
- conjunctive normal form (CNF) is conjunction of disjunctions
- 3-CNF is conjunction of disjunctions with 3 literals: $\bigwedge_i (a_i \vee b_i \vee c_i)$
- disjunctive normal form (DNF) is disjunction of conjunctions

Theorem

for every formula φ there is CNF ψ, 3-CNF χ and DNF η such that $\varphi \equiv \psi \equiv \chi \equiv \eta$

Remarks

- translation from formula to CNF can result in exponential blowup
- Tseitin's transformation is linear and produces equisatisfiable formula

Satisfiability (SAT)

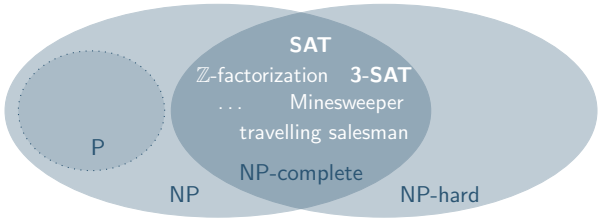
instance: propositional formula φ
question: is φ satisfiable?

3-Satisfiability (3-SAT)

instance: propositional formula φ in 3-CNF
question: is φ satisfiable?

Theorem

SAT and 3-SAT are NP-complete problems



- 1 million $ prize money awarded for solution to **P** $=^?$ **NP**

Outline

Approach

- most state-of-the-art SAT solvers use variation of Davis - Putnam - Logemann - Loveland (DPLL) procedure (1962)
- DPLL is sound and complete backtracking-based search algorithm
- can be described abstractly by transition system (Nieuwenhuis, Oliveras, Tinelli 2006)

Definition (Abstract DPLL)

- decision literal is annotated literal l^d
- state is pair $M \parallel F$ for
  - list M of (decision) literals — partial assignment
  - formula F in CNF
- transition rules

$$M \parallel F \quad \Longrightarrow \quad M' \parallel F' \quad \text{or} \quad \text{FailState}$$

Definition (DPLL Transition Rules)

- unit propagation $\quad M \parallel F,\ C \vee l \ \Longrightarrow\ M\, l \parallel F,\ C \vee l$
  if $M \vDash \neg C$ and l is undefined in M
- pure literal $\quad M \parallel F \ \Longrightarrow\ M\, l \parallel F$
  if l occurs in F but l^c does not occur in F, and l is undefined in M
- decide $\quad M \parallel F \ \Longrightarrow\ M\, l^d \parallel F$
  if l or l^c occurs in F, and l is undefined in M
- backtrack $\quad M\, l^d\, N \parallel F, C \ \Longrightarrow\ M\, l^c \parallel F, C$
  if $M\, l^d\, N \vDash \neg C$ and N contains no decision literals
- fail $\quad M \parallel F, C \ \Longrightarrow\ \text{FailState}$
  if $M \vDash \neg C$ and M contains no decision literals
- backjump $\quad M\, l^d\, N \parallel F, C \ \Longrightarrow\ M\, l' \parallel F, C$
  if $M\, l^d\, N \vDash \neg C$ and $\exists$ clause $C' \vee l'$ such that
  - $F, C \vDash C' \vee l'$ — backjump clause
  - $M \vDash \neg C'$ and l' is undefined in M, and l' or l'^c occurs in F or in $M\, l^d\, N$

Example

$\varphi = (\bar{1} \vee \bar{2}) \wedge (2 \vee 3) \wedge (\bar{1} \vee \bar{3} \vee 4) \wedge (2 \vee \bar{3} \vee \bar{4}) \wedge (1 \vee 4)$

$$
\begin{array}{rll}
 & \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \\
\Longrightarrow & 1^d \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{decide} \\
\Longrightarrow & 1^d\, \bar{2} \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\, \bar{2}\, 3 \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1^d\, \bar{2}\, 3\, 4 \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & \bar{1} \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{backtrack} \\
\Longrightarrow & \bar{1}\, 4 \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{unit propagate} \\
\Longrightarrow & \bar{1}\, 4\, 3^d \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{decide} \\
\Longrightarrow & \bar{1}\, 4\, 3^d\, 2 \parallel \bar{1} \vee \bar{2},\ 2 \vee 3,\ \bar{1} \vee \bar{3} \vee 4,\ 2 \vee \bar{3} \vee \bar{4},\ 1 \vee 4 & \text{unit propagate}
\end{array}
$$

Example (Backjump)

$\varphi = (\bar{1} \vee 2) \wedge (\bar{1} \vee \bar{3} \vee 4 \vee 5) \wedge (\bar{2} \vee \bar{4} \vee \bar{5}) \wedge (4 \vee \bar{5}) \wedge (\bar{4} \vee 5)$

$$
\begin{array}{rll}
 & \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \\
\Longrightarrow & 1^d \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\, 2 \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{unit propagate} \\
\Longrightarrow & 1^d\, 2\, 3^d \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\, 2\, 3^d\, 4^d \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{decide} \\
\Longrightarrow & 1^d\, 2\, 3^d\, 4^d\, \bar{5} \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{unit propagate} \\
\Longrightarrow & 1^d\, 2\, 3^d\, \bar{4} \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{backtrack} \\
\Longrightarrow & 1^d\, 2\, 3^d\, \bar{4}\, 5 \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{unit propagate} \\
\Longrightarrow & \ldots & \text{backtrack} \\
\Longrightarrow & 1^d\, 2\, \bar{3} \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 & \text{backjump} \\
\Longrightarrow^{+} & 1^d\, 2\, \bar{3}\, \bar{4}\, \bar{5} \parallel \bar{1} \vee 2,\ \bar{1} \vee \bar{3} \vee 4 \vee 5,\ \bar{2} \vee \bar{4} \vee \bar{5},\ 4 \vee \bar{5},\ \bar{4} \vee 5 &
\end{array}
$$

Decisions 1 and 3 are incompatible: $\varphi \vDash \bar{1} \vee \bar{3}$

Definition

basic DPLL $\mathcal{B}$ consists of unit propagation, decide, fail, and backjump

Properties

if $\| F \Longrightarrow_{\mathcal{B}}^{*} M \| F'$ then

- $F = F'$
- M does not contain complementary literals
- literals in M are distinct
- length of M is bounded by number of atoms

Lemma (Model Entailment)

Suppose $\| F \Longrightarrow_{\mathcal{B}}^{*} M \| F$ *such that*

- $M = M_0\, l_1^d\, M_1\, l_2^d\, M_2 \ldots l_k^d\, M_k$ *and*
- *there are no decision literals in* M_i.

decisions imply all other literals in M

Then $F,\ l_1, \ldots, l_i \vDash M_i$ *for all* $0 \leqslant i \leqslant k$.

Theorem (Termination)

for any formula F *there are no infinite derivations*

$$\| F \quad \Longrightarrow_{\mathcal{B}} \quad S_1 \quad \Longrightarrow_{\mathcal{B}} \quad S_2 \quad \Longrightarrow_{\mathcal{B}} \quad \ldots$$

missing literals in M

Proof.

- for list of distinct literals M, define $a(M) = n - |M|$ where
  - n is number of propositional variables
  - $|M|$ is length of M
- measure state $M_0\, l_1^d\, M_1\, l_2^d\, M_2 \ldots l_k^d\, M_k \| F$ by tuple

$$(a(M_0),\ a(M_1),\ \ldots,\ a(M_k),\ \underbrace{\infty, \ldots, \infty}_{n-k})$$

- compare tuples lexicographically by extension of $>_{\mathbb{N}}$ with ∞ maximal
- every transition step decreases measure ■

Example (Revisited for termination)

$\varphi = (\overline{1} \vee \overline{2}) \wedge (2 \vee 3) \wedge (\overline{1} \vee \overline{3} \vee 4) \wedge (2 \vee \overline{3} \vee \overline{4}) \wedge (1 \vee 4)$

	$\| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$		$(n, \infty, \ldots)$
$\Longrightarrow$	$1^d \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	decide	$(n, n, \infty, \ldots)$
$\Longrightarrow$	$1^d\, \overline{2} \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n, n-1, \infty, \ldots)$
$\Longrightarrow$	$1^d\, \overline{2}\, 3 \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n, n-2, \infty, \ldots)$
$\Longrightarrow$	$1^d\, \overline{2}\, 3\, 4 \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n, n-3, \infty, \ldots)$
$\Longrightarrow$	$\overline{1} \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	backtrack	$(n-1, \infty, \ldots)$
$\Longrightarrow$	$\overline{1}\, 4 \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n-2, \infty, \ldots)$
$\Longrightarrow$	$\overline{1}\, 4\, 3^d \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	decide	$(n-2, n, \infty, \ldots)$
$\Longrightarrow$	$\overline{1}\, 4\, 3^d\, 2 \| \overline{1} \vee \overline{2},\ 2 \vee 3,\ \overline{1} \vee \overline{3} \vee 4,\ \ldots$	unit propagate	$(n-2, n-1, \infty, \ldots)$

Observations used in proof

- decide replaces ∞ by n
- unit propagate, backtrack, and backjump replace m by $m - 1$

Consider maximal derivation with final state S_n:

$$\| F \quad \Longrightarrow_{\mathcal{B}} \quad S_1 \quad \Longrightarrow_{\mathcal{B}} \quad S_2 \quad \Longrightarrow_{\mathcal{B}} \quad \ldots \quad \Longrightarrow_{\mathcal{B}} \quad S_n$$

Theorem

if $S_n = \text{FailState}$ *then* F *is unsatisfiable*

Proof.

- must have $\| F \Longrightarrow_{\mathcal{B}}^{*} M \| F \Longrightarrow_{\text{fail}} \text{FailState}$ such that M contains no decision literals and $M \vDash \neg C$ for some C in F
- by Model Entailment Lemma $F \vDash M$, so $F \vDash \neg C$
- also have $F \vDash C$ because C is in F, so F is unsatisfiable ■

Theorem

if $S_n = M \| F'$ *then* F *is satisfiable and* $M \vDash F$

Proof.

- have $F = F'$
- S_n is final, so all literals of F are defined in M (otherwise decide applicable)
- $\nexists$ clause C in F such that $M \vDash \neg C$ (otherwise backjump or fail applicable)
- so M satisfies F ($M \vDash F$) ■

Outline

Fact
most SAT solvers require input to be in CNF

Remarks

- transforming formula to equivalent CNF can cause exponential blowup
- transforming formula into equisatisfiable CNF is possible in linear time

Definition
formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \iff \psi \text{ is satisfiable}$$

Example

$$p \vee q \approx \top \qquad p \wedge \neg p \approx q \wedge \neg q \qquad p \wedge \neg p \not\approx p \wedge \neg q$$

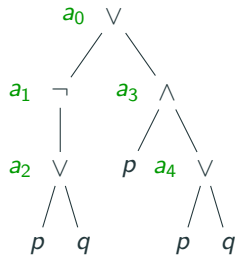
Example (Tseitin's Transformation)

- $\varphi = \neg(p \vee q) \vee (p \wedge (p \vee q))$
- use fresh propositional variable for every connective

  $a_0 : \neg(p \vee q) \vee (p \wedge (p \vee q))$ $\quad a_1 : \neg(p \vee q)$

  $a_2 : p \vee q$ $\quad a_3 : p \wedge (p \vee q)$

  $a_4 : p \vee q$



- $\varphi \approx \quad a_0 \wedge (a_0 \leftrightarrow a_1 \vee a_3) \wedge (a_1 \leftrightarrow \neg a_2) \wedge (a_2 \leftrightarrow p \vee q) \wedge$
  $(a_3 \leftrightarrow p \wedge a_4) \wedge (a_4 \leftrightarrow p \vee q)$
- every $\leftrightarrow$ subexpression can be replaced by at most three clauses:

$$\begin{aligned} a \leftrightarrow b \wedge c &\equiv (\neg a \vee b) \wedge (\neg a \vee c) \wedge (a \vee \neg b \vee \neg c) \\ a \leftrightarrow b \vee c &\equiv (\neg a \vee b \vee c) \wedge (a \vee \neg b) \wedge (a \vee \neg c) \\ a \leftrightarrow \neg b &\equiv (\neg a \vee \neg b) \wedge (a \vee b) \end{aligned}$$

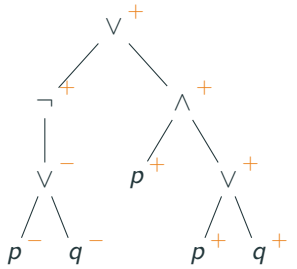
- common subexpressions can be shared

Observation
bi-implication $\leftrightarrow$ in Tseitin's transformation can be replaced by $\rightarrow$ or $\leftarrow$:
direction of implication $\rightarrow$ or $\leftarrow$ depends on polarity of subformula

Definition
for φ subformula occurrence of ψ

- let k be number of negations above φ in syntax tree of ψ
- polarity of φ is $+$ if k is even, and $-$ otherwise

Example



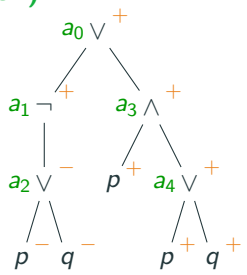
Example (Plaisted and Greenbaum's Transformation)

- $\varphi = \neg(p \vee q) \vee (p \wedge (p \vee q))$
- use fresh propositional variable for every connective

  $a_0 : \neg(p \vee q) \vee (p \wedge (p \vee q))$ $\quad a_1 : \neg(p \vee q)$

  $a_2 : p \vee q$ $\quad a_3 : p \wedge (p \vee q)$

  $a_4 : p \vee q$



- add $(a_i \rightarrow \dots)$ if polarity of a_i is positive and $(a_i \leftarrow \dots)$ if negative

  $\varphi \approx \quad a_0 \wedge (a_0 \rightarrow a_1 \vee a_3) \wedge (a_1 \rightarrow \neg a_2) \wedge (a_2 \leftarrow p \vee q) \wedge$
  $(a_3 \rightarrow p \wedge a_4) \wedge (a_4 \rightarrow p \vee q)$

- every $\leftarrow$ and $\rightarrow$ subexpression can be replaced by at most two clauses:

  $a \rightarrow b \wedge c \equiv (\neg a \vee b) \wedge (\neg a \vee c)$ $\quad a \leftarrow b \wedge c \equiv (a \vee \neg b \vee \neg c)$

  $a \rightarrow b \vee c \equiv (\neg a \vee b \vee c)$ $\quad a \leftarrow b \vee c \equiv (a \vee \neg b) \wedge (a \vee \neg c)$

  $a \rightarrow \neg b \equiv (\neg a \vee \neg b)$ $\quad a \leftarrow \neg b \equiv (a \vee b)$

SAT Solvers

Minisat, Glucose, CaDiCaL, Glu_VC, Plingeling, MapleLRB LCM, MapleCOMPSPS, Riss, Lingeling, Treengeling, CryptoMiniSat, abcdSAT, Dimetheus, Kiel, MapleCOMSPS, Rsat, SWDiA5BY, BlackBox, SWDiA5BY, pprobSAT, glueSplit_clasp, BalancedZ, SApperloT, PeneLoPe, MXC, ROKKminisat, MiniSat_HACK_999ED, ZENN, CSHCrandMC, MiniGolf, march_rw, sattime2011, mphasesat64, sparrow2011, pmcSAT, CSHCpar8, gluebit_clasp, clasp, precosat, gNovelty, SATzilla, SatELite, Score2SAT, YalSAT, tch glucose3, ...

SAT Competition

- annual competition for different tracks (main, parallel, no-limit, ...)
- increasing set of benchmarks from industry, mathematics, cryptography, ...
- standardized input format DIMACS and proof format DRAT

`http://www.satcompetition.org/`

Minisat

- minimalistic open source solver (`http://minisat.se/` or apt, yum,...)

```
$ minisat test.sat result.txt
```

- web interface

Example (DIMACS)

formula $(x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee \neg x_1) \wedge (\neg x_1 \vee x_2 \vee x_4)$ can be expressed by

```
c a very simple example
p cnf 4 3
1 -3 0
2 3 -1 0
-1 2 4 0
```

The DIMACS format

- header `p cnf` *n m* specifies number of variables *n* and number of clauses *m*
- variables (atoms) are assumed to be $x_1, \dots, x_n$
- literal x_i is denoted `i` and literal $\neg x_i$ is denoted `-i`
- a clause is a list of literals terminated by `0`
- lines starting with `c` are considered comments

Z3

common open source SAT/SMT solver

`https://github.com/Z3Prover/z3`

Python interface to Z3

- pip package `https://pypi.org/project/z3-solver/`
  (or manual installation from project site above)
- API: `https://z3prover.github.io/api/html/namespacez3py.html`

Building formulas

- `True`, `False` — boolean constants
- `Bool(`*name*`)` — propositional variable named *name*
  (calling `Bool(`*name*`)` twice yields same variable)
- `And(`$a_1, \dots, a_n$`)` — conjunction with arbitrarily many arguments
- `Or(`$a_1, \dots, a_n$`)` — disjunction with arbitrarily many arguments
- `Not(`a`)` — negation
- `Implies(`a, b`)` — implication
- `Xor(`a, b`)` — exclusive or

Solving formulas

- `Solver()` create new solver object
- `Solver.add(`$\varphi_1, \ldots, \varphi_n$`)` require constraints $\varphi_1, \ldots, \varphi_n$ to be true
- `Solver.check()` check for satisfiability
- `Solver.model()` returns valuation (after successful call of `check`)

Moreover ...

- `simplify(`φ`)` simplifies formula φ
- `Solver.statistics()` is map of solving statistics

Example

```
from z3 import *

foo = Bool('foo') # create variables named 'foo', 'bar', 'qax'
bar = Bool('bar')
qax = Bool('qax')

phi = Or(foo, And(bar, Xor(foo, Not(qax)), True), False)
print(phi) # Or(foo, And(bar, Xor(foo, Not(qax)), True), False)
psi = simplify(phi)
print(psi) # Or(foo, And(bar, foo == qax))

solver = Solver()
solver.add(psi) # assert that psi should be true
solver.add(Implies(foo, qax), Or(bar, foo)) # assert something else

print(solver) # [Or(foo, And(bar, foo == qax)), Implies(foo, qax), ...]
result = solver.check() # check for satisfiability

if result:
  model = solver.model() # get valuation
  print(model[foo], model[bar], model[qax]) # False True False
```


Example (Minesweeper)

x_1		x_2	
x_3	x_4	x_5	x_6
x_7		x_8	
x_9	x_{10}	x_{11}	

SAT Encoding

- variable x_i for each unknown cell i, $v(x_i) = \mathsf{T}$ iff cell i has mine
- constraints for every hint (number in grid)

1 $(x_2 \vee x_5 \vee x_6) \wedge ((\neg x_2 \wedge \neg x_5) \vee (\neg x_2 \wedge \neg x_6) \vee (\neg x_5 \wedge \neg x_6))$

8 $x_3 \wedge x_4 \wedge x_5 \wedge x_7 \wedge x_8 \wedge x_9 \wedge x_{10} \wedge x_{11}$

3 $((x_5 \wedge x_6 \wedge x_8) \vee (x_5 \wedge x_6 \wedge x_{11}) \vee (x_5 \wedge x_8 \wedge x_{11}) \vee (x_6 \wedge x_8 \wedge x_{11})) \wedge (\neg x_5 \vee \neg x_6 \vee \neg x_8 \vee \neg x_{11})$

2 $x_8 \wedge x_{11}$

3 $\bigvee_{1 \leqslant i,j \leqslant 5, i \neq j} \neg x_i \wedge \neg x_j \qquad \bigvee_{1 \leqslant i,j,k \leqslant 5, i \neq j, i \neq k, j \neq k} x_i \wedge x_j \wedge x_k$

DPLL

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Solving SAT and SAT Modulo Theories: From an Abstract Davis-Putnam-Logemann-Loveland Procedure to DPLL(T).
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Application Examples

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