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# SAT and SMT Solving

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# Outline

- Summary of Last Week
- Unsatisfiable Cores
- Application: FPGA Routing
- Algorithm by Fu and Malik
- Unsatisfiable Cores in Practice

# Maximum Satisfiability

Consider CNF formulas $\chi$ and $\varphi$ as sets of clauses such that $\chi$ is satisfiable.

**Definitions**

- maxSAT$(\varphi)$ is maximal $\sum_{C \in \varphi} \overline{v}(C)$ for valuation $v$
- pmaxSAT$(\varphi, \chi)$ is maximal $\sum_{C \in \varphi} \overline{v}(C)$ for valuation $v$ with $v(\chi) = \mathsf{T}$

**Definitions**

given weights $w_C \in \mathbb{Z}$ for all $C \in \varphi$,

- $\mathrm{maxSAT}_w(\varphi)$ is maximal $\sum_{C \in \varphi} w_C \cdot \overline{v}(C)$ for valuation $v$?
- $\mathrm{pmaxSAT}_w(\varphi, \chi)$ is maximal $\sum_{C \in \varphi} w_C \cdot \overline{v}(C)$ for valuation $v$ with $v(\chi) = \mathsf{T}$

**Definition**

minUNSAT$(\varphi)$ is minimal $\sum_{C \in \varphi} \overline{v}(\neg C)$ for valuation $v$

**Lemma**

$$|\varphi| = |\mathrm{minUNSAT}(\varphi)| + |\mathrm{maxSAT}(\varphi)|$$

# Branch & Bound

## Idea

- gets list of clauses $\varphi$ as input return minUNSAT($\varphi$)
- explores assignments in depth-first search

---

```
function BnB($\varphi$, UB)
  $\varphi$ = simp($\varphi$)
  if $\varphi$ contains only empty clauses then
      return #empty($\varphi$)
  if #empty($\varphi$) $\geqslant$ UB then
      return UB
  x = selectVariable($\varphi$)
  UB' = min(UB, BnB($\varphi_x$, UB))
  return min(UB', BnB($\varphi_{\overline{x}}$, UB'))
```

---

## Theorem

$\mathrm{BnB}(\varphi, |\varphi|) = \mathrm{minUNSAT}(\varphi)$

# Binary Search

## Idea

- gets list of clauses $\varphi$ as input and returns minUNSAT($\varphi$)
- repeatedly call SAT solver in binary search fashion

## Definitions

- cardinality constraint is

  $$\sum_{x \in X} x \bowtie N$$

  where $\bowtie$ is $=$, $<$, $>$, $\leqslant$, or $\geqslant$, $X$ is set of propositional variables, and $N \in \mathbb{N}$
- valuation $v$ satisfies $\sum_{x \in X} x \bowtie N$ iff $k \bowtie N$
  where $k$ is number of variables $x \in X$ such that $v(x) = \mathsf{T}$

## Remark

cardinality constraints are expressible in CNF

## Algorithm (Binary Search)

```
function BinarySearch($\{C_1, \ldots, C_m\}$)
  $\varphi := \{C_1 \vee b_1, \ldots, C_m \vee b_m\}$
  return search($\varphi$,0,m)
```

$b_1, \ldots, b_m$ are fresh variables

```
function search($\varphi$, L, U)
  if L $\geqslant$ U then
    return U
  mid $:= \lfloor \frac{U+L}{2} \rfloor$
  if SAT($\varphi \wedge$ CNF($\sum_{i=1}^{m} b_i \leqslant$ mid)) then
    return search($\varphi$, L, mid)
  else
    return search($\varphi$, mid + 1, U)
```

## Theorem

$\texttt{BinarySearch}(\psi) = \mathsf{minUNSAT}(\psi)$

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$$\varphi = \{\neg x, \qquad x \vee z, \qquad \neg y \vee \neg z, \qquad x, \qquad y \vee \neg z\}$$

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## Remark

SUC is always minimal unsatisfiable core

# Example

$\varphi = \{C_1, \dots, C_6\}$ is unsatisfiable

$C_1: x_1 \lor \neg x_3$

$C_2: x_2$

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$C_4: \neg x_2 \lor \neg x_3$

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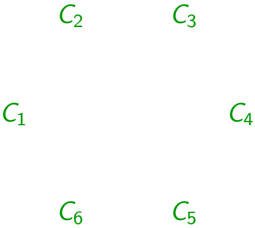

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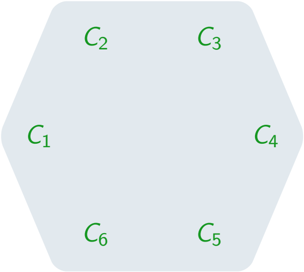

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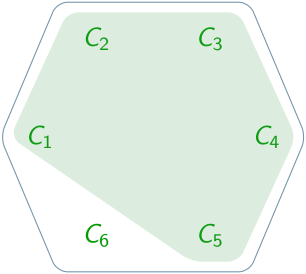

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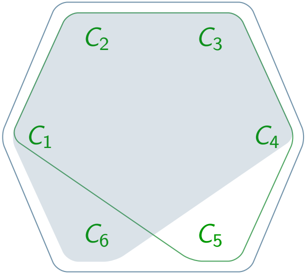

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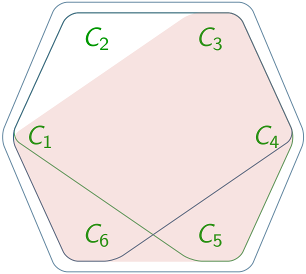

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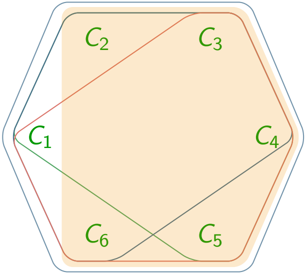

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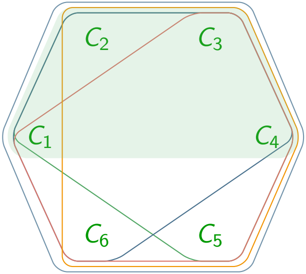

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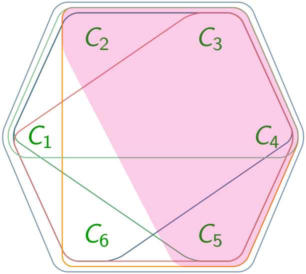

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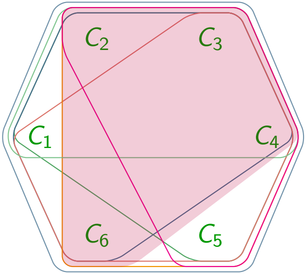

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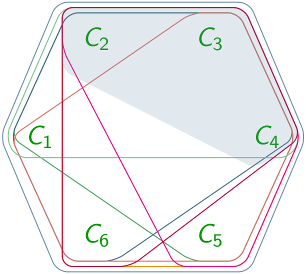

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## Idea

- repeatedly pick clause $C$ from $\varphi$ and check satisfiability:
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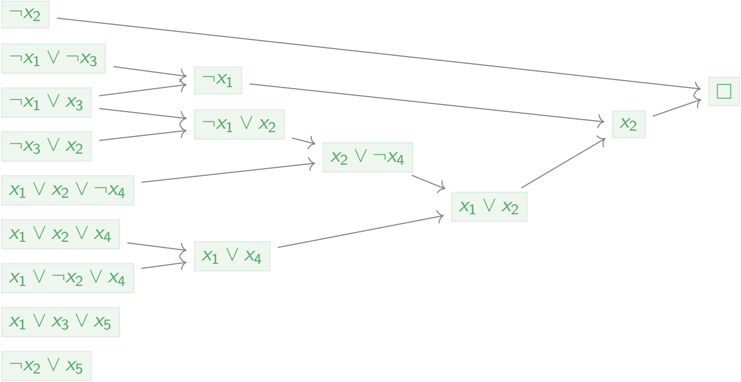

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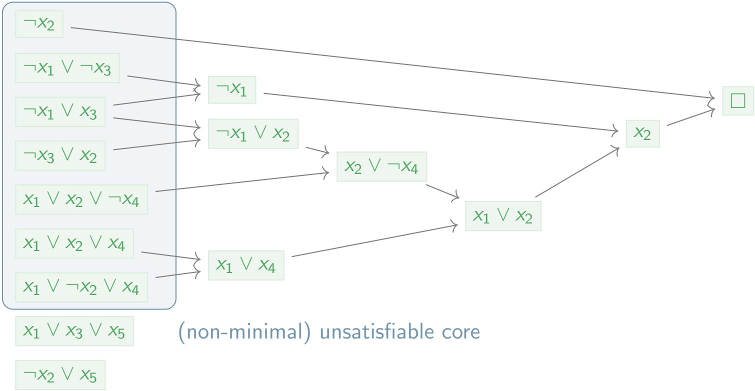

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**Definition (Resolution Graph)**

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### Notation

- $\mathit{Reach}_G(C)$ is set of nodes reachable from $C$ in $G$

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5. there are no other edges.

## Remark

- if $\varphi$ is unsatisfiable then sequence of resolution steps can derive $\square$ because resolution is complete proof method
- so resolution graph exists

## Notation

- $\mathit{Reach}_G(C)$ is set of nodes reachable from $C$ in $G$
- $\mathit{Reach}^E_G(C)$ is set of edges reachable from $C$ in $G$

Assume $\varphi$ is unsatisfiable.

### Definition (Resolution Graph)

directed acyclic graph $G = (V, E)$ is resolution graph for set of clauses $\varphi$ if

1. $V = V_i \uplus V_c$ is set of clauses and $V_i = \varphi$,
2. $V_i$ nodes have no incoming edges, initial nodes
3. there is exactly one node $\Box$ without outgoing edges,
4. $\forall C \in V_c \; \exists$ edges $D \to C$, $D' \to C$ such that $C$ is resolvent of $D$ and $D'$, and
5. there are no other edges.

### Remark

- if $\varphi$ is unsatisfiable then sequence of resolution steps can derive $\Box$ because resolution is complete proof method
- so resolution graph exists

### Notation

- $Reach_G(C)$ is set of nodes reachable from $C$ in $G$
- $Reach_G^E(C)$ is set of edges reachable from $C$ in $G$
- $\overline{N}$ is $V \setminus N$ for any set of nodes $N$

```
Algorithm minUnsatCore($\varphi$)
Input:   unsatisfiable formula $\varphi$
Output:  minimal unsatisfiable core of $\varphi$
  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
  while $\exists$ unmarked clause in $V_i$ do
      $C \leftarrow$ unmarked clause in $V_i$
      if SAT($\overline{Reach_G(C)}$) then          ▷ subgraph without $C$ satisfiable?
          mark $C$                                  ▷ $C$ is UC member
      else
          build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
          $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\square$                 ▷ restrict to nodes with path to $\square$
  return $V_i$
```

**Algorithm** minUnsatCore($\varphi$)

**Input:** unsatisfiable formula $\varphi$
**Output:** minimal unsatisfiable core of $\varphi$

```
build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
while $\exists$ unmarked clause in $V_i$ do
    $C \leftarrow$ unmarked clause in $V_i$
    if SAT($\overline{Reach_G(C)}$) then                         ▷ subgraph without $C$ satisfiable?
        mark $C$                                                  ▷ $C$ is UC member
    else
        build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
        $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
        $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
        $G \leftarrow (V_i \cup V_c, E)$
        $G \leftarrow G|_\square$                                 ▷ restrict to nodes with path to $\square$
return $V_i$
```

```
**Algorithm** minUnsatCore($\varphi$)

**Input:**     unsatisfiable formula $\varphi$
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  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
  **while** $\exists$ unmarked clause in $V_i$ **do**
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      **if** SAT($\overline{Reach_G(C)}$) **then**                ▷ subgraph without $C$ satisfiable?
          mark $C$                                             ▷ $C$ is UC member
      **else**
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          $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\square$                                    ▷ restrict to nodes with path to $\square$
  return $V_i$
```

```
Algorithm minUnsatCore($\varphi$)

Input:    unsatisfiable formula $\varphi$
Output:   minimal unsatisfiable core of $\varphi$
  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
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      mark $C$                                   ▷ $C$ is UC member
    else
      build resolution graph $G' = (V'_i \uplus V'_c, E')$ for $\overline{Reach_G(C)}$
      $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V'_c \cup (V_c \setminus Reach_G(C))$
      $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
      $G \leftarrow (V_i \cup V_c, E)$
      $G \leftarrow G|_\Box$                      ▷ restrict to nodes with path to $\Box$
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```
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**Input:** unsatisfiable formula $\varphi$
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        $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
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**Input:** unsatisfiable formula $\varphi$

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```
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    if SAT($\overline{Reach_G(C)}$) then            ▷ subgraph without $C$ satisfiable?
      mark $C$                                      ▷ $C$ is UC member
    else
      build resolution graph $G' = (V'_i \uplus V'_c, E')$ for $\overline{Reach_G(C)}$
      $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V'_c \cup (V_c \setminus Reach_G(C))$
      $E \leftarrow E' \cup (E \setminus Reach^E_G(C))$
      $G \leftarrow (V_i \cup V_c, E)$
      $G \leftarrow G|_\Box$                         ▷ restrict to nodes with path to $\Box$
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```
**Algorithm** minUnsatCore($\varphi$)
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**Algorithm** minUnsatCore($\varphi$)

**Input:** unsatisfiable formula $\varphi$
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    if SAT($\overline{Reach_G(C)}$) then                    ▷ subgraph without $C$ satisfiable?
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```

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**Algorithm** minUnsatCore($\varphi$)
**Input:**   unsatisfiable formula $\varphi$
**Output:**  minimal unsatisfiable core of $\varphi$
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  **while** $\exists$ unmarked clause in $V_i$ **do**
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      **if** SAT($\overline{Reach_G(C)}$) **then**        ▷ subgraph without $C$ satisfiable?
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      **else**
          build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
          $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\square$                       ▷ restrict to nodes with path to $\square$
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```

```
Algorithm minUnsatCore($\varphi$)

Input:   unsatisfiable formula $\varphi$
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      $C \leftarrow$ unmarked clause in $V_i$
      if SAT($\overline{Reach_G(C)}$) then                      ▷ subgraph without $C$ satisfiable?
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          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\square$                             ▷ restrict to nodes with path to $\square$
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```

```
Algorithm minUnsatCore($\varphi$)

Input:    unsatisfiable formula $\varphi$
Output:   minimal unsatisfiable core of $\varphi$
  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
  while $\exists$ unmarked clause in $V_i$ do
      $C \leftarrow$ unmarked clause in $V_i$
      if SAT($\overline{Reach_G(C)}$) then            ▷ subgraph without $C$ satisfiable?
          mark $C$                                     ▷ $C$ is UC member
      else
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          $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\Box$                        ▷ restrict to nodes with path to $\Box$
  return $V_i$
```

```
Algorithm minUnsatCore($\varphi$)

Input:    unsatisfiable formula $\varphi$
Output:   minimal unsatisfiable core of $\varphi$
  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
  while $\exists$ unmarked clause in $V_i$ do
      $C \leftarrow$ unmarked clause in $V_i$
      if $\mathrm{SAT}(\overline{Reach_G(C)})$ then          ▷ subgraph without $C$ satisfiable?
          mark $C$                                         ▷ $C$ is UC member
      else
          build resolution graph $G' = (V_i' \uplus V_c', E')$ for $\overline{Reach_G(C)}$
          $V_i \leftarrow V_i \setminus \{C\}$ and $V_c \leftarrow V_c' \cup (V_c \setminus Reach_G(C))$
          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\Box$                          ▷ restrict to nodes with path to $\Box$
  return $V_i$
```

**Algorithm** minUnsatCore($\varphi$)

```
Input:   unsatisfiable formula $\varphi$
Output:  minimal unsatisfiable core of $\varphi$
  build resolution graph $G = (V_i \uplus V_c, E)$ for $\varphi$
  while $\exists$ unmarked clause in $V_i$ do
      $C \leftarrow$ unmarked clause in $V_i$
      if SAT($\overline{Reach_G(C)}$) then          ▷ subgraph without $C$ satisfiable?
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          $E \leftarrow E' \cup (E \setminus Reach_G^E(C))$
          $G \leftarrow (V_i \cup V_c, E)$
          $G \leftarrow G|_\Box$                     ▷ restrict to nodes with path to $\Box$
  return $V_i$
```
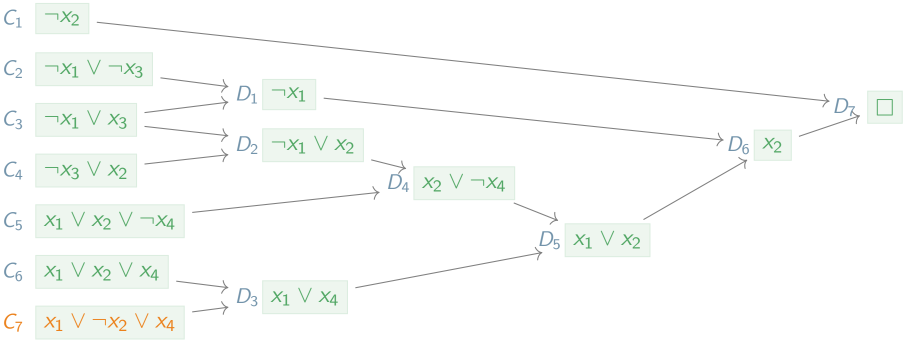

**Theorem**

*if $\varphi$ unsatisfiable then minUnsatCore($\varphi$) is minimal unsatisfiable core of $\varphi$*

## Example



$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$C_7$ $x_1 \vee \neg x_2 \vee x_4$

$D_1$ $\neg x_1$

$D_2$ $\neg x_1 \vee x_2$

$D_3$ $x_1 \vee x_4$

$D_4$ $x_2 \vee \neg x_4$

$D_5$ $x_1 \vee x_2$

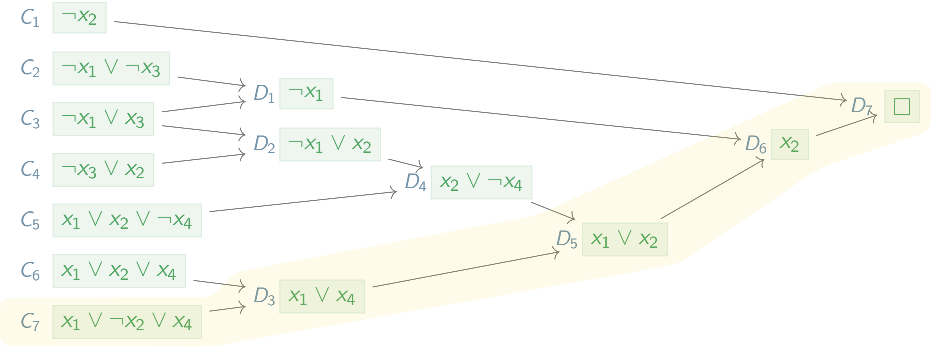
$D_6$ $x_2$

$D_7$ $\Box$

**minUnsatCore($\varphi$)**

- pick $C_7$

## Example



$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$C_7$ $x_1 \vee \neg x_2 \vee x_4$

$D_1$ $\neg x_1$

$D_2$ $\neg x_1 \vee x_2$

$D_3$ $x_1 \vee x_4$

$D_4$ $x_2 \vee \neg x_4$

$D_5$ $x_1 \vee x_2$

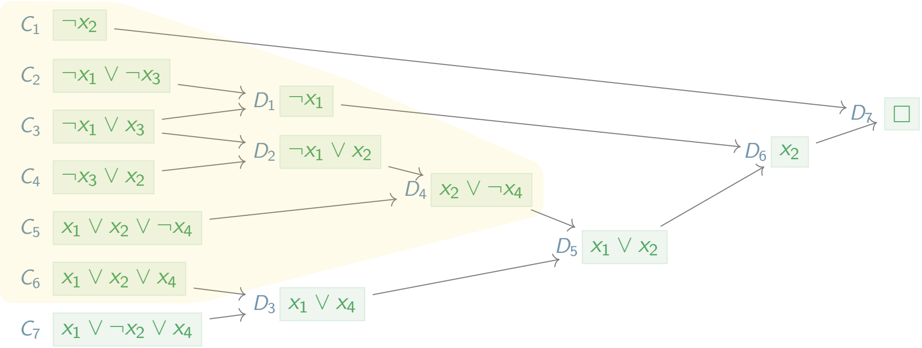
$D_6$ $x_2$

$D_7$ $\square$

**minUnsatCore($\varphi$)**

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$

# Example



$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$C_7$ $x_1 \vee \neg x_2 \vee x_4$

$D_1$ $\neg x_1$

$D_2$ $\neg x_1 \vee x_2$

$D_3$ $x_1 \vee x_4$

$D_4$ $x_2 \vee \neg x_4$

$D_5$ $x_1 \vee x_2$

$D_6$ $x_2$

$D_7$ $\square$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$

# Example

$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$D_1$ $\neg x_1$

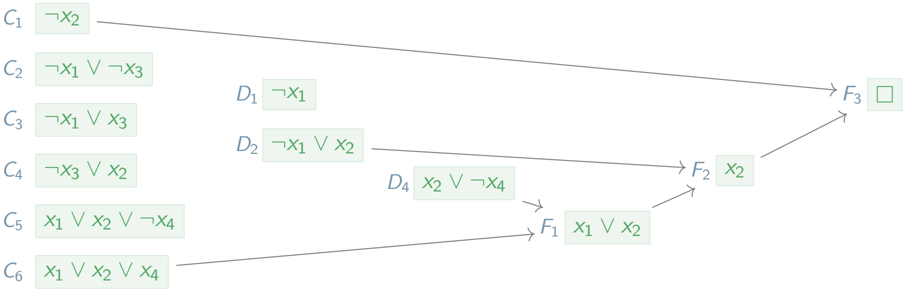
$D_2$ $\neg x_1 \vee x_2$

$D_4$ $x_2 \vee \neg x_4$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$
- check SAT($\overline{Reach_G(C_7)}$)

# Example



$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$D_1$ $\neg x_1$

$D_2$ $\neg x_1 \vee x_2$

$D_4$ $x_2 \vee \neg x_4$

$F_1$ $x_1 \vee x_2$

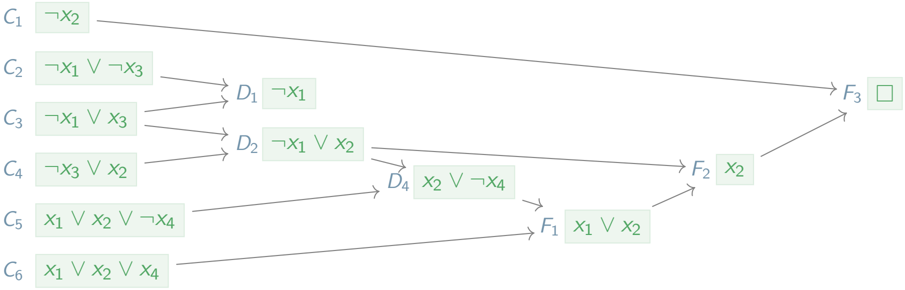
$F_2$ $x_2$

$F_3$ $\square$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$
- check SAT($\overline{Reach_G(C_7)}$)
- unsatisfiable: get new resolution graph $G_7$ for $\varphi \cup \{D_1, D_2, D_4\}$

# Example



$C_1$ $\neg x_2$

$C_2$ $\neg x_1 \vee \neg x_3$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$D_1$ $\neg x_1$

$D_2$ $\neg x_1 \vee x_2$

$D_4$ $x_2 \vee \neg x_4$

$F_1$ $x_1 \vee x_2$

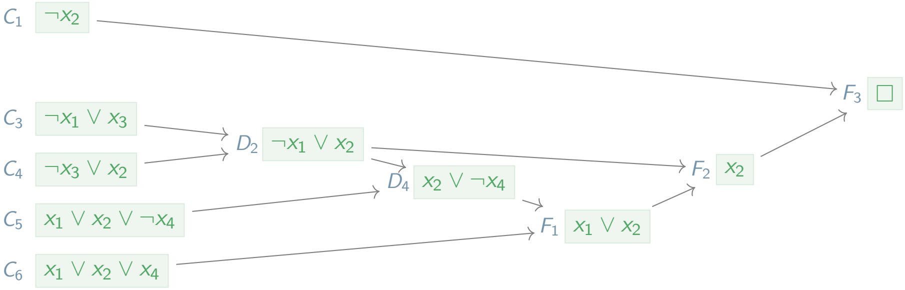
$F_2$ $x_2$

$F_3$ $\square$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$
- check SAT($\overline{Reach_G(C_7)}$)
- unsatisfiable: get new resolution graph $G_7$ for $\varphi \cup \{D_1, D_2, D_4\}$
- construct resolution graph $G'$ for $\varphi$ by adding edges from $G$ to $G_7$

# Example



$C_1$ $\neg x_2$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$D_2$ $\neg x_1 \vee x_2$

$D_4$ $x_2 \vee \neg x_4$

$F_1$ $x_1 \vee x_2$

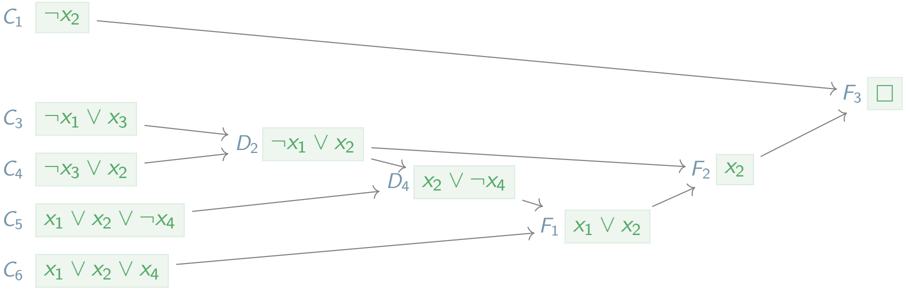
$F_2$ $x_2$

$F_3$ $\square$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$
- check SAT($\overline{Reach_G(C_7)}$)
- unsatisfiable: get new resolution graph $G_7$ for $\varphi \cup \{D_1, D_2, D_4\}$
- construct resolution graph $G'$ for $\varphi$ by adding edges from $G$ to $G_7$
- set $G$ to $G'$ restricted to nodes with path to $\square$

# Example



$C_1$ $\neg x_2$

$C_3$ $\neg x_1 \vee x_3$

$C_4$ $\neg x_3 \vee x_2$

$C_5$ $x_1 \vee x_2 \vee \neg x_4$

$C_6$ $x_1 \vee x_2 \vee x_4$

$D_2$ $\neg x_1 \vee x_2$

$D_4$ $x_2 \vee \neg x_4$

$F_1$ $x_1 \vee x_2$

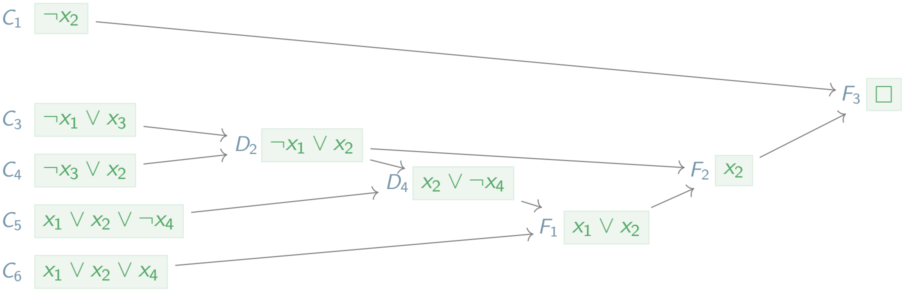
$F_2$ $x_2$

$F_3$ $\Box$

## minUnsatCore($\varphi$)

- pick $C_7$
- $Reach_G(C_7) = \{C_7, D_3, D_5, D_6, D_7\}$ $\overline{Reach_G(C_7)} = \{C_1, \ldots, C_6, D_1, D_2, D_4\}$
- check SAT($\overline{Reach_G(C_7)}$)
- unsatisfiable: get new resolution graph $G_7$ for $\varphi \cup \{D_1, D_2, D_4\}$
- construct resolution graph $G'$ for $\varphi$ by adding edges from $G$ to $G_7$
- set $G$ to $G'$ restricted to nodes with path to $\Box$
- after 5 more loop iterations: return $\{C_1, C_3, \ldots, C_6\}$

# Example



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re-use relevant resolvents:
fewer steps to $\Box$

# Application: FPGA Routing

## Field Programmable Gate Arrays (FPGAs)

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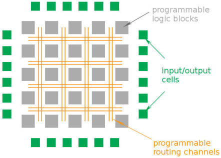

# Application: FPGA Routing

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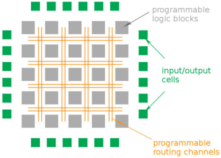

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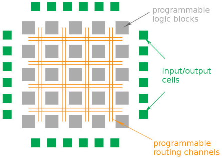
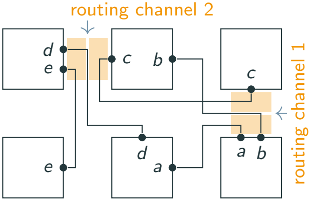


## Example (Encoding Routing Requirements)

- consider connections $a, b, c, d, e$ of 2 bits each



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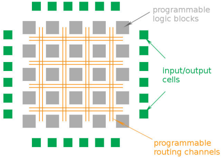

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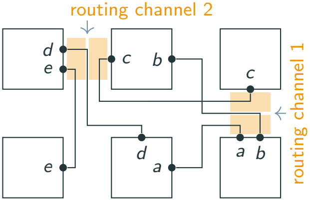

$$a_0 \vee a_1$$
$$b_0 \vee b_1$$
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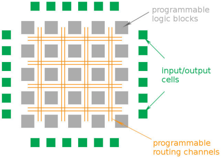

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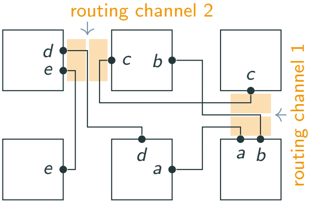

$a_0 \vee a_1$

$b_0 \vee b_1$

$c_0 \vee c_1$

$d_0 \vee d_1$

$e_0 \vee e_1$

$\neg a_0 \vee \neg b_0$

$\neg a_0 \vee \neg c_0$

$\neg b_0 \vee \neg c_0$

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$\neg c_0 \vee \neg d_0$

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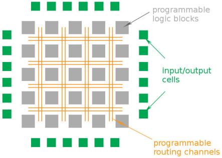

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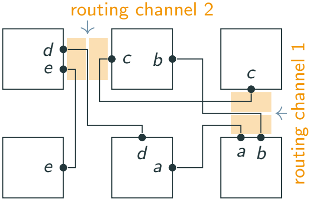

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$a_0 \lor a_1$

$b_0 \lor b_1$

$c_0 \lor c_1$

$d_0 \lor d_1$

$e_0 \lor e_1$

$\neg a_0 \lor \neg b_0$

$\neg a_0 \lor \neg c_0$

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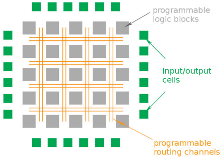

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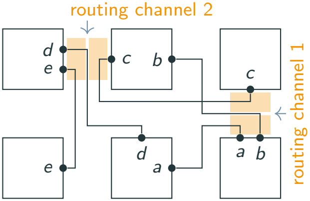


$a_0 \vee a_1$ ●
$b_0 \vee b_1$ ●
$c_0 \vee c_1$ ●
$d_0 \vee d_1$
$e_0 \vee e_1$

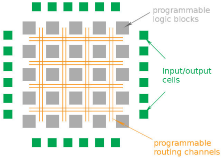
$\neg a_0 \vee \neg b_0$ ●
$\neg a_0 \vee \neg c_0$ ●
$\neg b_0 \vee \neg c_0$ ●
$\neg a_1 \vee \neg b_1$ ●
$\neg a_1 \vee \neg c_1$ ●
$\neg b_1 \vee \neg c_1$ ●

$\neg c_0 \vee \neg d_0$
$\neg c_0 \vee \neg e_0$
$\neg d_0 \vee \neg e_0$
$\neg c_1 \vee \neg d_1$
$\neg c_1 \vee \neg e_1$
$\neg d_1 \vee \neg e_1$

$UC_1$: channel 1 capacity exceeded

# Application: FPGA Routing

## Field Programmable Gate Arrays (FPGAs)



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$d_0 \vee d_1$ ●

$e_0 \vee e_1$ ●

$\neg a_0 \vee \neg b_0$ ●

$\neg a_0 \vee \neg c_0$ ●

$\neg b_0 \vee \neg c_0$ ●

$\neg a_1 \vee \neg b_1$ ●

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$\neg b_1 \vee \neg c_1$ ●

$\neg c_0 \vee \neg d_0$ ●

$\neg c_0 \vee \neg e_0$ ●

$\neg d_0 \vee \neg e_0$ ●

$\neg c_1 \vee \neg d_1$ ●

$\neg c_1 \vee \neg e_1$ ●

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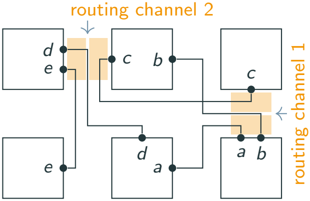

$UC_1$: channel 1 capacity exceeded

$UC_2$: channel 2 capacity exceeded

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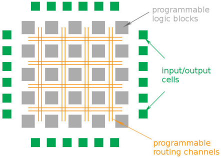

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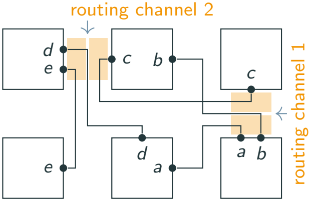

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$b_0 \vee b_1$ 🟠🔴
$c_0 \vee c_1$ 🟠🟢🔴
$d_0 \vee d_1$ 🟢🔴
$e_0 \vee e_1$ 🟢🔴

$\neg a_0 \vee \neg b_0$ 🟠
$\neg a_0 \vee \neg c_0$ 🟠🔴
$\neg b_0 \vee \neg c_0$ 🟠🔴
$\neg a_1 \vee \neg b_1$ 🟠🔴
$\neg a_1 \vee \neg c_1$ 🟠
$\neg b_1 \vee \neg c_1$ 🟠

$\neg c_0 \vee \neg d_0$ 🟢
$\neg c_0 \vee \neg e_0$ 🟢
$\neg d_0 \vee \neg e_0$ 🟢🔴
$\neg c_1 \vee \neg d_1$ 🟢🔴
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$UC_1$: channel 1 capacity exceeded
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$UC_3$: $c$ is overconstrained

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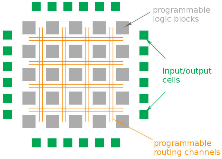

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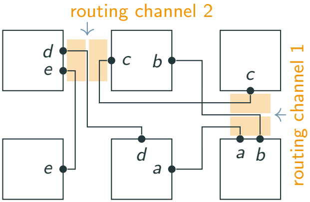

$a_0 \vee a_1$ 🟠 🔴 🔵
$b_0 \vee b_1$ 🟠 🔴 🔵
$c_0 \vee c_1$ 🟠 🟢 🔴 🔵
$d_0 \vee d_1$ 🟢 🔴 🔵
$e_0 \vee e_1$ 🟢 🔴 🔵

$\neg a_0 \vee \neg b_0$ 🟠 🔵
$\neg a_0 \vee \neg c_0$ 🟠 🔴
$\neg b_0 \vee \neg c_0$ 🟠 🔴
$\neg a_1 \vee \neg b_1$ 🟠 🔴
$\neg a_1 \vee \neg c_1$ 🟠 🔵
$\neg b_1 \vee \neg c_1$ 🟠 🔵

$\neg c_0 \vee \neg d_0$ 🟢 🔵
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$\neg d_0 \vee \neg e_0$ 🟢 🔴
$\neg c_1 \vee \neg d_1$ 🟢 🔴
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$UC_4$: $c$ is overconstrained

# Outline

- Summary of Last Week
- Unsatisfiable Cores
- Application: FPGA Routing
- Algorithm by Fu and Malik
- Unsatisfiable Cores in Practice

# Bounds for Maximum Satisfiability

consider CNF formula $\varphi = C_1 \wedge \cdots \wedge C_m$

## Definition

blocked formula is $\varphi_B = (C_1 \vee b_1) \wedge \cdots \wedge (C_m \vee b_m)$ for fresh variables $b_1, \ldots, b_m$

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**Lemma (Lower Bound)**

*if* $v$ *satisfies* $\varphi_B$ *and* $B_\mathsf{T} = \{b_i \mid v(b_i) = \mathsf{T}\}$ *then* $\mathsf{maxSAT}(\varphi) \geqslant m - |B_\mathsf{T}|$

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## Example (Upper Bound)

$\neg x_3 \vee \neg x_4$ $\quad$ $\neg x_3 \vee x_4$

$x_3$

$x_1$ $\quad$ $\neg x_1 \vee \neg x_3$

$\neg x_1 \vee \neg x_2$ $\quad$ $\neg x_1 \vee x_2$

$x_7$

$\neg x_7 \vee x_8$ $\quad$ $\neg x_1 \vee x_8$

$\neg x_7 \vee \neg x_8 \vee x_6$ $\quad$ $\neg x_9 \vee x_2$

$\neg x_7 \vee \neg x_8 \vee \neg x_6$

$x_4 \vee x_5$ $\quad$ $x_1 \vee \neg x_5 \vee x_6$

$x_5 \vee \neg x_6$

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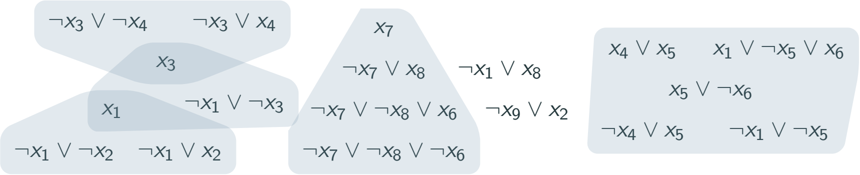

$\neg x_3 \vee \neg x_4$ $\quad$ $\neg x_3 \vee x_4$

$x_3$

$x_1$ $\quad$ $\neg x_1 \vee \neg x_3$

$\neg x_1 \vee \neg x_2$ $\quad$ $\neg x_1 \vee x_2$

$x_7$

$\neg x_7 \vee x_8$

$\neg x_7 \vee \neg x_8 \vee x_6$

$\neg x_7 \vee \neg x_8 \vee \neg x_6$

$\neg x_1 \vee x_8$

$\neg x_9 \vee x_2$

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$x_5 \vee \neg x_6$

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unsatisfiable cores

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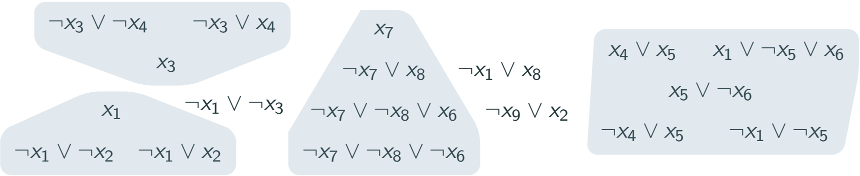


$\neg x_3 \vee \neg x_4$ $\quad$ $\neg x_3 \vee x_4$ $\quad$ $x_3$

$x_1$ $\quad$ $\neg x_1 \vee \neg x_2$ $\quad$ $\neg x_1 \vee x_2$

$\neg x_1 \vee \neg x_3$

$x_7$ $\quad$ $\neg x_7 \vee x_8$ $\quad$ $\neg x_7 \vee \neg x_8 \vee x_6$ $\quad$ $\neg x_7 \vee \neg x_8 \vee \neg x_6$

$\neg x_1 \vee x_8$ $\quad$ $\neg x_9 \vee x_2$

$x_4 \vee x_5$ $\quad$ $x_1 \vee \neg x_5 \vee x_6$ $\quad$ $x_5 \vee \neg x_6$ $\quad$ $\neg x_4 \vee x_5$ $\quad$ $\neg x_1 \vee \neg x_5$

4 disjoint unsatisfiable cores

# Bounds for Maximum Satisfiability

consider CNF formula $\varphi = C_1 \wedge \cdots \wedge C_m$

**Definition**

blocked formula is $\varphi_B = (C_1 \vee b_1) \wedge \cdots \wedge (C_m \vee b_m)$ for fresh variables $b_1, \dots, b_m$

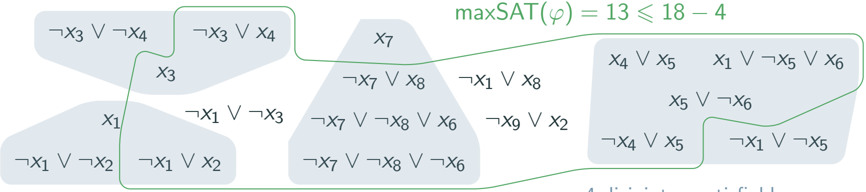
**Lemma (Lower Bound)**

*if $v$ satisfies $\varphi_B$ and $B_\mathsf{T} = \{b_i \mid v(b_i) = \mathsf{T}\}$ then* $\mathsf{maxSAT}(\varphi) \geqslant m - |B_\mathsf{T}|$

**Lemma (Upper Bound)**

*if $\varphi$ contains $k$ disjoint unsatisfiable cores then* $\mathsf{maxSAT}(\varphi) \leqslant m - k$

**Example (Upper Bound)**



$\mathsf{maxSAT}(\varphi) = 13 \leqslant 18 - 4$

$\neg x_3 \vee \neg x_4$ $\quad \neg x_3 \vee x_4$ $\quad x_3$

$x_1$ $\quad \neg x_1 \vee \neg x_2$ $\quad \neg x_1 \vee x_2$

$\neg x_1 \vee \neg x_3$

$x_7$ $\quad \neg x_7 \vee x_8$ $\quad \neg x_7 \vee \neg x_8 \vee x_6$ $\quad \neg x_7 \vee \neg x_8 \vee \neg x_6$

$\neg x_1 \vee x_8$ $\quad \neg x_9 \vee x_2$

$x_4 \vee x_5$ $\quad x_1 \vee \neg x_5 \vee x_6$ $\quad x_5 \vee \neg x_6$ $\quad \neg x_4 \vee x_5$ $\quad \neg x_1 \vee \neg x_5$

4 disjoint unsatisfiable cores

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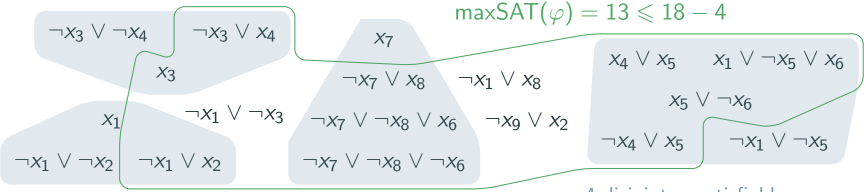
## Lemma (Upper Bound)

*if $\varphi$ contains $k$ disjoint unsatisfiable cores then* $\mathrm{maxSAT}(\varphi) \leqslant m - k$

## Example (Upper Bound)

must miss at least one clause from every core!



$\mathrm{maxSAT}(\varphi) = 13 \leqslant 18 - 4$

$\neg x_3 \vee \neg x_4$ $\quad$ $\neg x_3 \vee x_4$ $\quad$ $x_3$

$x_1$ $\quad$ $\neg x_1 \vee \neg x_2$ $\quad$ $\neg x_1 \vee x_2$

$\neg x_1 \vee \neg x_3$

$x_7$ $\quad$ $\neg x_7 \vee x_8$ $\quad$ $\neg x_7 \vee \neg x_8 \vee x_6$ $\quad$ $\neg x_7 \vee \neg x_8 \vee \neg x_6$

$\neg x_1 \vee x_8$ $\quad$ $\neg x_9 \vee x_2$

$x_4 \vee x_5$ $\quad$ $x_1 \vee \neg x_5 \vee x_6$ $\quad$ $x_5 \vee \neg x_6$ $\quad$ $\neg x_4 \vee x_5$ $\quad$ $\neg x_1 \vee \neg x_5$

4 disjoint unsatisfiable cores

# Algorithm by Fu and Malik

**Idea**

- maxsat valuation must make at least one clause in unsatisfiable core false

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- maxsat valuation must make at least one clause in unsatisfiable core false
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## Definition (Partial minUNSAT)

pminUNSAT$(\chi, \varphi)$ is minimal $\sum_{C \in \varphi} \overline{v}(\neg C)$ for valuation $v$ with $v(\chi) = \mathsf{T}$

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- maxsat valuation must make at least one clause in unsatisfiable core false
- while there exists (minimal) unsatisfiable core:

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**Definition (Partial** minUNSAT**)**

pminUNSAT$(\chi, \varphi)$ is minimal $\sum_{C \in \varphi} \overline{v}(\neg C)$ for valuation $v$ with $v(\chi) = \mathsf{T}$

**Lemma**

$$|\varphi| = \mathsf{pminUNSAT}(\chi, \varphi) + \mathsf{pmaxSAT}(\chi, \varphi)$$

# Example

$\chi$: $\neg x_1 \lor x_3$   $\neg x_7 \lor x_2$   $x_7 \lor x_2$   $x_1 \lor \neg x_2$

$\varphi$: $\neg x_1 \lor \neg x_2$   $\neg x_1 \lor x_2$   $\neg x_1 \lor x_7$   $x_1$
$\neg x_3 \lor x_4$   $x_3$   $\neg x_3 \lor \neg x_4$   $x_4 \lor x_5$
$\neg x_4 \lor x_5$   $x_1 \lor \neg x_5 \lor x_6$   $x_5 \lor \neg x_6$   $x_7$
$\neg x_7 \lor x_8$   $\neg x_7 \lor \neg x_8 \lor x_6$   $\neg x_7 \lor \neg x_8 \lor \neg x_6$   $\neg x_1 \lor \neg x_3$

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \lor x_3$ | $\neg x_7 \lor x_2$ | $x_7 \lor x_2$ | $x_1 \lor \neg x_2$ |
| $\varphi$: | $\neg x_1 \lor \neg x_2$ | $\neg x_1 \lor x_2$ | $\neg x_1 \lor x_7$ | $x_1$ |
| | $\neg x_3 \lor x_4$ | $x_3$ | $\neg x_3 \lor \neg x_4$ | $x_4 \lor x_5$ |
| | $\neg x_4 \lor x_5$ | $x_1 \lor \neg x_5 \lor x_6$ | $x_5 \lor \neg x_6$ | $x_7$ |
| | $\neg x_7 \lor x_8$ | $\neg x_7 \lor \neg x_8 \lor x_6$ | $\neg x_7 \lor \neg x_8 \lor \neg x_6$ | $\neg x_1 \lor \neg x_3$ |

- unsatisfiable core: $\neg x_1 \lor \neg x_2$, $\neg x_1 \lor x_2$, $x_1$

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \vee x_3$ | $\neg x_7 \vee x_2$ | $x_7 \vee x_2$ | $x_1 \vee \neg x_2$ |
| $\varphi$: | $\neg x_1 \vee \neg x_2 \vee b_1$ | $\neg x_1 \vee x_2 \vee b_2$ | $\neg x_1 \vee x_7$ | $x_1 \vee b_3$ |
| | $\neg x_3 \vee x_4$ | $x_3$ | $\neg x_3 \vee \neg x_4$ | $x_4 \vee x_5$ |
| | $\neg x_4 \vee x_5$ | $x_1 \vee \neg x_5 \vee x_6$ | $x_5 \vee \neg x_6$ | $x_7$ |
| | $\neg x_7 \vee x_8$ | $\neg x_7 \vee \neg x_8 \vee x_6$ | $\neg x_7 \vee \neg x_8 \vee \neg x_6$ | $\neg x_1 \vee \neg x_3$ |

- unsatisfiable core: $\neg x_1 \vee \neg x_2, \ \neg x_1 \vee x_2, \ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$

## Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \vee x_3$ | $\neg x_7 \vee x_2$ | $x_7 \vee x_2$ | $x_1 \vee \neg x_2$ |
| $\varphi$: | $\neg x_1 \vee \neg x_2 \vee b_1$ | $\neg x_1 \vee x_2 \vee b_2$ | $\neg x_1 \vee x_7$ | $x_1 \vee b_3$ |
| | $\neg x_3 \vee x_4$ | $x_3$ | $\neg x_3 \vee \neg x_4$ | $x_4 \vee x_5$ |
| | $\neg x_4 \vee x_5$ | $x_1 \vee \neg x_5 \vee x_6$ | $x_5 \vee \neg x_6$ | $x_7$ |
| | $\neg x_7 \vee x_8$ | $\neg x_7 \vee \neg x_8 \vee x_6$ | $\neg x_7 \vee \neg x_8 \vee \neg x_6$ | $\neg x_1 \vee \neg x_3$ |

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \vee x_3$ | $\neg x_7 \vee x_2$ | $x_7 \vee x_2$ | $x_1 \vee \neg x_2$ |
| $\varphi$: | $\neg x_1 \vee \neg x_2 \vee b_1$ | $\neg x_1 \vee x_2 \vee b_2$ | $\neg x_1 \vee x_7$ | $x_1 \vee b_3$ |
| | $\neg x_3 \vee x_4 \vee c_1$ | $x_3 \vee c_2$ | $\neg x_3 \vee \neg x_4 \vee c_3$ | $x_4 \vee x_5$ |
| | $\neg x_4 \vee x_5$ | $x_1 \vee \neg x_5 \vee x_6$ | $x_5 \vee \neg x_6$ | $x_7$ |
| | $\neg x_7 \vee x_8$ | $\neg x_7 \vee \neg x_8 \vee x_6$ | $\neg x_7 \vee \neg x_8 \vee \neg x_6$ | $\neg x_1 \vee \neg x_3$ |

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$

# Example

$\chi$: $\neg x_1 \vee x_3$ $\quad \neg x_7 \vee x_2$ $\quad x_7 \vee x_2$ $\quad x_1 \vee \neg x_2$

$\varphi$: $\neg x_1 \vee \neg x_2 \vee b_1$ $\quad \neg x_1 \vee x_2 \vee b_2$ $\quad \neg x_1 \vee x_7$ $\quad x_1 \vee b_3$
$\neg x_3 \vee x_4 \vee c_1$ $\quad x_3 \vee c_2$ $\quad \neg x_3 \vee \neg x_4 \vee c_3$ $\quad x_4 \vee x_5$
$\neg x_4 \vee x_5$ $\quad x_1 \vee \neg x_5 \vee x_6$ $\quad x_5 \vee \neg x_6$ $\quad x_7$
$\neg x_7 \vee x_8$ $\quad \neg x_7 \vee \neg x_8 \vee x_6$ $\quad \neg x_7 \vee \neg x_8 \vee \neg x_6$ $\quad \neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$

# Example

$\chi$: $\neg x_1 \vee x_3$ $\quad \neg x_7 \vee x_2$ $\quad x_7 \vee x_2$ $\quad x_1 \vee \neg x_2$

$\varphi$: $\neg x_1 \vee \neg x_2 \vee b_1$ $\quad \neg x_1 \vee x_2 \vee b_2$ $\quad \neg x_1 \vee x_7$ $\quad x_1 \vee b_3$
$\neg x_3 \vee x_4 \vee c_1$ $\quad x_3 \vee c_2$ $\quad \neg x_3 \vee \neg x_4 \vee c_3$ $\quad x_4 \vee x_5$
$\neg x_4 \vee x_5$ $\quad x_1 \vee \neg x_5 \vee x_6$ $\quad x_5 \vee \neg x_6$ $\quad x_7 \vee d_1$
$\neg x_7 \vee x_8 \vee d_2$ $\quad \neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$ $\quad \neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$ $\quad \neg x_1 \vee \neg x_3$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \vee x_3$ | $\neg x_7 \vee x_2$ | $x_7 \vee x_2$ | $x_1 \vee \neg x_2$ |
| $\varphi$: | $\neg x_1 \vee \neg x_2 \vee b_1$ | $\neg x_1 \vee x_2 \vee b_2$ | $\neg x_1 \vee x_7$ | $x_1 \vee b_3$ |
| | $\neg x_3 \vee x_4 \vee c_1$ | $x_3 \vee c_2$ | $\neg x_3 \vee \neg x_4 \vee c_3$ | $x_4 \vee x_5$ |
| | $\neg x_4 \vee x_5$ | $x_1 \vee \neg x_5 \vee x_6$ | $x_5 \vee \neg x_6$ | $x_7 \vee d_1$ |
| | $\neg x_7 \vee x_8 \vee d_2$ | $\neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$ | $\neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$ | $\neg x_1 \vee \neg x_3$ |

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \text{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \text{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \text{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$

# Example

$$\begin{array}{lllll}
\chi: & \neg x_1 \vee x_3 & \neg x_7 \vee x_2 & x_7 \vee x_2 & x_1 \vee \neg x_2 \\
\varphi: & \neg x_1 \vee \neg x_2 \vee b_1 & \neg x_1 \vee x_2 \vee b_2 & \neg x_1 \vee x_7 & x_1 \vee b_3 \\
 & \neg x_3 \vee x_4 \vee c_1 & x_3 \vee c_2 & \neg x_3 \vee \neg x_4 \vee c_3 & x_4 \vee x_5 \\
 & \neg x_4 \vee x_5 & x_1 \vee \neg x_5 \vee x_6 & x_5 \vee \neg x_6 & x_7 \vee d_1 \\
 & \neg x_7 \vee x_8 \vee d_2 & \neg x_7 \vee \neg x_8 \vee x_6 \vee d_3 & \neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4 & \neg x_1 \vee \neg x_3 \vee e_1
\end{array}$$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$
  $\chi = \chi \cup \mathrm{CNF}(e_1 = 1)$
  $cost = 4$

# Example

$\chi$: $\neg x_1 \vee x_3$ $\quad \neg x_7 \vee x_2$ $\quad x_7 \vee x_2$ $\quad x_1 \vee \neg x_2$

$\varphi$: $\neg x_1 \vee \neg x_2 \vee b_1$ $\quad \neg x_1 \vee x_2 \vee b_2$ $\quad \neg x_1 \vee x_7$ $\quad x_1 \vee b_3$
$\neg x_3 \vee x_4 \vee c_1$ $\quad x_3 \vee c_2$ $\quad \neg x_3 \vee \neg x_4 \vee c_3$ $\quad x_4 \vee x_5$
$\neg x_4 \vee x_5$ $\quad x_1 \vee \neg x_5 \vee x_6$ $\quad x_5 \vee \neg x_6$ $\quad x_7 \vee d_1$
$\neg x_7 \vee x_8 \vee d_2$ $\quad \neg x_7 \vee \neg x_8 \vee x_6 \vee d_3$ $\quad \neg x_7 \vee \neg x_8 \vee \neg x_6 \vee d_4$ $\quad \neg x_1 \vee \neg x_3 \vee e_1$

- unsatisfiable core: $\neg x_1 \vee \neg x_2,\ \neg x_1 \vee x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \vee x_4,\ x_3, \neg x_3 \vee \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \vee x_8,\ \neg x_7 \vee \neg x_8 \vee x_6,\ \neg x_7 \vee \neg x_8 \vee \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \vee x_3,\ \neg x_7 \vee x_2,\ x_7 \vee x_2,\ x_1 \vee \neg x_2,\ \neg x_1 \vee \neg x_3$
  $\chi = \chi \cup \mathrm{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \lor x_3$ | $\neg x_7 \lor x_2$ | $x_7 \lor x_2$ | $x_1 \lor \neg x_2$ |
| $\varphi$: | $\neg x_1 \lor \neg x_2 \lor b_1$ | $\neg x_1 \lor x_2 \lor b_2$ | $\neg x_1 \lor x_7$ | $x_1 \lor b_3$ |
| | $\neg x_3 \lor x_4 \lor c_1$ | $x_3 \lor c_2$ | $\neg x_3 \lor \neg x_4 \lor c_3$ | $x_4 \lor x_5$ |
| | $\neg x_4 \lor x_5$ | $x_1 \lor \neg x_5 \lor x_6$ | $x_5 \lor \neg x_6$ | $x_7 \lor d_1$ |
| | $\neg x_7 \lor x_8 \lor d_2$ | $\neg x_7 \lor \neg x_8 \lor x_6 \lor d_3$ | $\neg x_7 \lor \neg x_8 \lor \neg x_6 \lor d_4$ | $\neg x_1 \lor \neg x_3 \lor e_1$ |

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \lor x_4,\ x_3, \neg x_3 \lor \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \lor x_8,\ \neg x_7 \lor \neg x_8 \lor x_6,\ \neg x_7 \lor \neg x_8 \lor \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \lor x_3,\ \neg x_7 \lor x_2,\ x_7 \lor x_2,\ x_1 \lor \neg x_2,\ \neg x_1 \lor \neg x_3$
  $\chi = \chi \cup \mathrm{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable
- $\mathrm{pminUNSAT}(\chi, \varphi) = 4$

# Example

| | | | | |
|---|---|---|---|---|
| $\chi$: | $\neg x_1 \lor x_3$ | $\neg x_7 \lor x_2$ | $x_7 \lor x_2$ | $x_1 \lor \neg x_2$ |
| $\varphi$: | $\neg x_1 \lor \neg x_2 \lor b_1$ | $\neg x_1 \lor x_2 \lor b_2$ | $\neg x_1 \lor x_7$ | $x_1 \lor b_3$ |
| | $\neg x_3 \lor x_4 \lor c_1$ | $x_3 \lor c_2$ | $\neg x_3 \lor \neg x_4 \lor c_3$ | $x_4 \lor x_5$ |
| | $\neg x_4 \lor x_5$ | $x_1 \lor \neg x_5 \lor x_6$ | $x_5 \lor \neg x_6$ | $x_7 \lor d_1$ |
| | $\neg x_7 \lor x_8 \lor d_2$ | $\neg x_7 \lor \neg x_8 \lor x_6 \lor d_3$ | $\neg x_7 \lor \neg x_8 \lor \neg x_6 \lor d_4$ | $\neg x_1 \lor \neg x_3 \lor e_1$ |

- unsatisfiable core: $\neg x_1 \lor \neg x_2,\ \neg x_1 \lor x_2,\ x_1$
  $\chi = \chi \cup \mathrm{CNF}(b_1 + b_2 + b_3 = 1)$
  $cost = 1$
- unsatisfiable core: $\neg x_3 \lor x_4,\ x_3, \neg x_3 \lor \neg x_4$
  $\chi = \chi \cup \mathrm{CNF}(c_1 + c_2 + c_3 = 1)$
  $cost = 2$
- unsatisfiable core: $x_7,\ \neg x_7 \lor x_8,\ \neg x_7 \lor \neg x_8 \lor x_6,\ \neg x_7 \lor \neg x_8 \lor \neg x_6$
  $\chi = \chi \cup \mathrm{CNF}(d_1 + d_2 + d_3 + d_4 = 1)$
  $cost = 3$
- unsatisfiable core: $\neg x_1 \lor x_3,\ \neg x_7 \lor x_2,\ x_7 \lor x_2,\ x_1 \lor \neg x_2,\ \neg x_1 \lor \neg x_3$
  $\chi = \chi \cup \mathrm{CNF}(e_1 = 1)$
  $cost = 4$
- satisfiable: $v(x_1) = v(x_2) = v(x_3) = v(x_5) = v(x_7) = \mathrm{T}$ and $v(x_i) = \mathrm{F}$ otherwise
- pminUNSAT$(\chi, \varphi) = 4$ and pmaxSAT$(\chi, \varphi) = 12$

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Output:   pminUNSAT$(\chi, \varphi)$
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      for $C \in UC \cap \varphi$ do                               ▷ loop over soft clauses in core
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**Theorem**

FuMalik($\chi, \varphi$) = pminUNSAT($\chi, \varphi$)

# Unsatisfiable Cores in z3

```
from z3 import *

x1,x2,x3 = Bool("x1"), Bool("x2"), Bool("x3")
phi = [ Or(Not(x1), Not(x2)), Or(Not(x1), x2),\
  Or(Not(x1), x3), x1, Or(Not(x3), x2)]

solver = Solver()
solver.set(unsat_core=True)

# assert clauses in phi with names phi0 ... phi4
for i,c in enumerate(phi):
  solver.assert_and_track(c, "phi" + str(i))

if solver.check() == z3.unsat:
  uc = solver.unsat_core()
  print(uc) # [phi0, phi1, phi3]
```

# Literature

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